VIII/4. $Av = \lambda v$ és $A^* w = \mu w$.

Ha v és w nem $\perp$ $\Rightarrow$ $\overline{\lambda} = \mu$.

<u>Biz:</u>

$$\langle Av, w \rangle = \langle v, A^* w \rangle$$

$$\langle \lambda v, w \rangle \qquad \langle v, \mu w \rangle$$

$$\overline{\lambda} \langle v, w \rangle \qquad \mu \langle v, w \rangle.$$

$\operatorname{Ker} A^* = (\operatorname{Im} A)^\perp$.

$A^* v = 0$ $\qquad$ $v \perp \operatorname{Im} A$ azaz $\langle v, Aw \rangle = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \langle A^* v, w \rangle$

Ha $A^* v = 0 \Rightarrow \langle A^* v, w \rangle = 0 \Rightarrow v \in (\operatorname{Im} A)^\perp$

$\langle A^* v, A^* v \rangle = 0 \Leftarrow \quad \forall w \in V \quad \Leftarrow$

$\Rightarrow A^* v = 0$ ✓

$\operatorname{Im} A^* = (\operatorname{Ker} A)^\perp$

$\to A \leadsto A^*$ $\qquad$ $\operatorname{Ker} A = (\operatorname{Im} A^*)^\perp$ $\quad / \perp$

$$(\operatorname{Ker} A)^\perp = (\operatorname{Im} A^*)^{\perp\perp} = \operatorname{Im} A^*.$$

(3) $AA^* = 0 \implies A = 0.$

$A^*A(v) = 0 \implies \langle A^*Av, w \rangle = 0 \quad \forall w$

$\qquad\qquad\qquad\qquad\qquad \langle Av, Aw \rangle = 0 \qquad v = w$

$\qquad\qquad\qquad\qquad\qquad\qquad \|Av\|^2 = 0 \implies Av = 0$

$A(v) = 0.$

$AA^* = 0 \qquad \Leftarrow \quad A^* = 0 \implies A = 0.$

---

(4) $A^*B = 0 \implies \operatorname{Im} A \cap \operatorname{Im} B = 0.$

$v \in \operatorname{Im} A \cap \operatorname{Im} B \qquad v = Au = Bw.$

$\langle v, v \rangle = \langle Au, Bw \rangle = \langle u, A^*Bw \rangle = 0.$

$\|v\|^2 \implies v = 0.$ $\qquad\qquad A^*B = 0$

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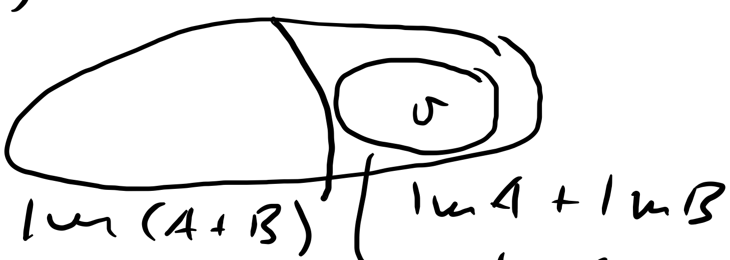
(5) $A^*B = BA^* = 0 \implies \operatorname{Im}(A+B) = \operatorname{Im} A + \operatorname{Im} B.$



$\operatorname{Im}(A+B) \subseteq \operatorname{Im} A + \operatorname{Im} B$

$(A+B)v = Av + Bv$

$\operatorname{Im}(A+B)$ ορθογ. συμπλήρωμα στο $\operatorname{Im} A + \operatorname{Im} B$

$-$ ker.

κάθε (ελέγ) : είναι 0. $\qquad v \in \operatorname{Im} A + \operatorname{Im} B$

$v \perp \operatorname{Im} A+B$

$\implies v = 0.$

$v \in \operatorname{Im} A + \operatorname{Im} B \qquad v = Au + Bw.$

$v \perp \operatorname{Im}(A+B) \qquad \langle v, (A+B)v_1 \rangle = 0 \quad \forall v_1 \in V.$

$\langle (A+B)^* v, v_1 \rangle = 0 \implies (A+B)^* v = 0.$

$(A+B)^*(Au + Bw) = 0.$

Tudjuk:
$A^* B = BA^* = 0.$

$$A^*Au + \underbrace{B^*Au}_{=0} + \underbrace{A^*Bw}_{=0} + B^*Bw = 0$$

$(A^*B)^* = B^*A = 0$

$$\underbrace{BA^*}_{=0} Au + \underbrace{BB^*Bw}_{} = B^*0 = 0.$$

$\langle BB^*Bw, Bw \rangle = \langle B^*Bw, B^*Bw \rangle$

$= 0 \qquad \|B^*Bw\|^2 = 0$

$\implies \boxed{B^*Bw = 0}$

$\langle B^*Bw, w \rangle = \langle Bw, Bw \rangle \implies \boxed{Bw = 0}$

$= 0$

$A^*Au = 0 \longrightarrow \langle A^*Au, u \rangle = \langle Au, Au \rangle$

$= 0 \qquad \implies Au = 0.$

$\implies v = 0.$

5. $(AB)^* = B^* A^* = BA$.

$A''B \iff A''B$.

---

6. Normális trafó sajátalterei páronként $\perp$

$\Longleftarrow$ igaz-e.

$[A] = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_n \end{bmatrix}$ $\quad Ab_i = \lambda_i b_i \quad b_i$ s.v.

λ-hoz tartozó sajátaltér $\langle b_i \mid \lambda_i = \lambda \rangle$.

$v = \sum \mu_i b_i \Rightarrow Av = \sum \lambda_i \mu_i b_i = \lambda v =$
$= \sum \lambda \mu_i b_i$.

$\iff \lambda_i \mu_i = \lambda \mu_i \ \forall i \iff \mu_i = 0$ ha $\lambda_i \neq \lambda$.

$b_1 \dots b_n$ ONB $\Rightarrow$ ezek a sajátalterek páronként $\perp$.

Megfordítás $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ nem igaz.

Igaz ha a s.alterek páronként $\perp$-ek ÉS diag-ható (azaz s.alt. dim. összege n)

↗ tér dim

$\Rightarrow$ normális.

7. $AB = BA$ $\overset{!}{\Longrightarrow}$ s. vektorok ugyanazok?

A, B normális $\underset{?}{\Longleftarrow}$

$\Rightarrow$ NEM $A = \mathrm{Id}$. $\forall B$ jó.

$\forall$ vektor s.v. Ha $B \not\ni E$ akkor van $\forall$ vektor s.vektor.

$\Leftarrow$ igaz. A $b_1, \ldots, b_n$ ONB-ben diag.

sajátaltér A-nak $\Rightarrow$ B-nek is

$\Rightarrow$ ugyanabban B is diagonális.

$\forall 2$ diag. mátrix felcs.

---

8. A normális $\iff \|Av\| = \|A^*v\| \ \forall v$.

$\|Av\| = \langle Av, Av \rangle = \langle v, A^*Av \rangle$

$\|A^*v\| = \langle v, AA^*v \rangle$

Ha $AA^* = A^*A$, akkor egyenlők.

$\Leftarrow$ $\mathbb{C}$ felett bilineáris fo-t sz. alak meghatározza.

$\langle v, AA^*v \rangle = \langle v, A^*Av \rangle \Rightarrow \langle v, AA^*w \rangle = \langle v, A^*Aw \rangle$ $\Rightarrow A^*A = AA^*$

(9.) $\perp$-tartó $\quad u \perp w \Rightarrow Au \perp Aw.$

$\langle u, w\rangle = 0 \Rightarrow \langle Au, Aw\rangle = 0$

$\qquad\qquad\qquad\qquad \langle u, A^*Aw\rangle.$

u fix. $\quad w \in \langle u\rangle^\perp \Rightarrow A^*Aw \in \langle u\rangle^\perp.$

$\Rightarrow \langle u\rangle^\perp \quad A^*A$-inv.

Lemma (ea.) $\quad U$ altér B-inv.

$\Rightarrow U^\perp \; B^*$-inv.

$\Rightarrow (A^*A)^*$-ra inv. $\left(\langle u\rangle^\perp\right)^\perp = \langle u\rangle$

$A^*A. \qquad \Rightarrow A^*Au \in \langle u\rangle$

$A^*Au = \lambda_u \cdot u$

$\forall u$ s.vektora A^*A-nak. Tfh u és w (F)

$A^*Au = \lambda_u u \qquad A^*Aw = \lambda_w w$

$A^*A(u+w) = \lambda_u u + \lambda_w w$

$\lambda(u+w) \overset{!!}{\Rightarrow} \lambda = \lambda_u \quad \lambda = \lambda_w$

(F) $\qquad \Rightarrow \lambda_u = \lambda_w.$

$b_1, \ldots, b_n$ (B)

$\lambda_1, \ldots, \lambda_n$

$\Rightarrow \lambda_1 = \ldots = \lambda_n$

$\boxed{A^*A = \lambda I}$

Ha $\lambda = 0$ $A^*A = 0 \Rightarrow A = 0 \Rightarrow \lambda \cdot E$
$\lambda = 0$. unitér.

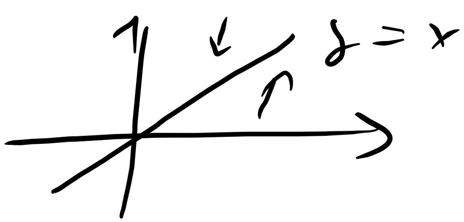
Ha $\lambda \neq 0$ A^*A ∀ s.é. valós, sőt ≥ 0.
$A^*A u = \lambda u$
$\langle A^*Au, u \rangle = \langle Au, Au \rangle = \|Au\|^2 \geq 0$ valós
$= \bar{\lambda} \langle u, u \rangle$ > 0 valós $\Rightarrow \bar{\lambda}$ pos. valós

$A^*A = \lambda E$ λ pos. valós $\Rightarrow \lambda \in \mathbb{R}$.

$\boxed{\frac{1}{\sqrt{\lambda}} A \text{ unitér}}$

10. A ortog. [illegible] $\Rightarrow A^2 = I$.
$AA^* = I$ $A^* = A$ ($= A^T$) $\Rightarrow A^2 = I$

Megfordítás nem igaz: nem $\perp$ vetítés ellenp.



$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$A^2 = I \Leftrightarrow A$ vetítés.
Az kell, hogy $\perp$ legyen $\operatorname{Im} A \perp \operatorname{Ker} A$

11. A normális $\Longleftrightarrow$ A^* pol-ja A-nak.

Ha $p(A) = A^* \Rightarrow p(A)A = A\,p(A)$

$\Rightarrow A$ normális.

Tfh A normális $\quad [A] = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$[A^*] = \begin{pmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_n} \end{pmatrix}$

$\exists$ interpolációs polinom

$p(\lambda_i) = \overline{\lambda_i} \Rightarrow p(A) = A^*$.

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12. M valós 2×2 normális.

$\mathbb{C}$ felett $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ kar pol valós együtthatós.

Vagy λ_1, λ_2 valós $\Rightarrow A$ szimm.

Vagy $\lambda_1 = \overline{\lambda_2}$ $\quad r\begin{bmatrix} \cos\alpha + i\sin\alpha & 0 \\ 0 & \cos\alpha - i\sin\alpha \end{bmatrix}$.

$|\lambda_1| = r$

$\rightsquigarrow r \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ azaz a forgatás skalárszorosai.

13.

$$A = U \cdot S$$

tetsz $\quad U^* = U^{-1} \qquad S^* = S.$

Ha A normális.

$$(A) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} =$$

$$\lambda_i = 0 \qquad \underset{U}{1} \cdot \overset{S}{0}$$

$$\lambda_i \neq 0 \qquad \underbrace{\frac{\lambda_i}{|\lambda_i|}}_{U} \; \underbrace{|\lambda_i|}_{S}$$

$$U \cdot S = S \cdot U.$$

$$|\lambda_i| = \sqrt{\lambda_i \overline{\lambda_i}}.$$

$$A = U \cdot S = S \cdot U$$

$$A^* = S^* U^* = S U^{-1}$$

$$A A^* = U \cdot S \cdot S U^{-1} = S \cdot S U \cdot U^{-1} = S^2$$

$$A^* A = S U^{-1} U S = S^2$$

$A = US \qquad U^* = U^{-1}, \quad S^* = S.$

$A^* = S^* U^* = S U^{-1}$

$A^*A = S U^{-1} U S = \underline{\underline{S^2}}.$

Kell: $\sqrt{A^*A}$.

A^*A önadj.: $(A^*A)^* = A^* A^{**} = A^*A.$

$\forall$ s. é. ≥ 0

$$[A^*A] = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad \lambda_i \geq 0 \qquad [S] = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}.$$

$V = V_1 \oplus V_2 \qquad b_1, \ldots, b_n$ ONB

$\langle b_i \mid \lambda_i = 0 \rangle \qquad \langle b_i \mid \lambda_i > 0 \rangle$

V_1-en $U = I$.

V_2-n S invertálható $\qquad U = A S^{-1}$ V_2-n.

$U^* = (S^{-1})^* A^*$

$$U^* U = (S^{-1})^* \underbrace{A^* A}_{S^2} S^{-1} = \underbrace{(S^{-1})^*}_{= S^{-1}} S = I.$$