MINIMAL ABELIAN VARIETIES OF ALGEBRAS, I

KEITH A. KEARNES, EMIL W. KISS, AND ÁGNES SZENDREI

Abstract. We show that any abelian variety that is not affine has a nontrivial strongly abelian subvariety. In later papers in this sequence we apply this result to the study of minimal abelian varieties.

1. Background

This paper concerns the classification of minimal varieties of algebras. Early work on this topic focused on determining the minimal subvarieties of known varieties (e.g., groups, rings, modules, lattices, etc). For a survey of these results, please consult [11]. This 1992 survey also contains the state of knowledge (at the time) of the problem of classifying all minimal locally finite varieties.

But shortly after the publication of [11], two groups of researchers (Szendrei on the one hand and Kearnes-Kiss-Valeriote on the other) independently classified the minimal, locally finite, abelian varieties of algebras. Multiple proofs of the classification theorem were discovered and presented in the papers [7, 8, 12, 13]. Those proofs start with the observation that each minimal locally finite variety contains a smallest nontrivial member, which must be finite, simple, and have no proper nontrivial subalgebras. Tame congruence theory assigns a number (a *type*) to any finite simple algebra: if abelian, the type must be **1** (the G-set type), or type **2** (the vector space type). The type **1**/type **2** case division is the main case division in the classification of minimal, locally finite, abelian varieties. Within each of these two cases there are subcases related to dimension and to the field associated to the vector space in type **2**. The full classification is accomplished by examining type **1** and type **2** simple algebras until one can isolate out and fully describe those which generate minimal varieties.

The problem of extending these results to varieties that are not locally finite has been considered, but only under additional very strong hypotheses. For example,

2010 *Mathematics Subject Classification.* Primary: 03C05; Secondary: 08A05, 08B15.

Key words and phrases. Abelian algebra, minimal variety, rectangular algebra, strongly abelian algebra, term condition.

This material is based upon work supported by the National Science Foundation grant no. DMS 1500254, the Hungarian National Foundation for Scientific Research (OTKA) grant no. K115518, and the National Research, Development and Innovation Fund of Hungary (NKFI) grant no. K128042.

in [1, Theorem 5.12] the classification of minimal abelian varieties is obtained under the assumption that the variety is idempotent and contains a nontrivial quasiaffine algebra. In [2, Corollary 2.10] this is generalized to eliminate the hypothesis that the variety contains a nontrivial quasiaffine algebra. The result here is a classification of arbitrary minimal, abelian, idempotent varieties

This is the first in a sequence of three papers in which we attempt to classify the minimal abelian varieties without any additional assumptions at all. We predict that a complete classification proof might evolve along these lines:

Goal 1. Show that minimal abelian varieties exist in two unrelated types, corresponding to the type **1**/type **2** case division observed in the locally finite setting. We propose that these types should be: strongly abelian minimal varieties as the extension of the type **1** case, and affine varieties as the extension of the type **2** case.

Goal 2. Classify the minimal affine varieties.

Goal 3. Classify the minimal strongly abelian varieties.

In this paper we accomplish Goal 1. Actually, we prove a stronger statement that does not involve minimality, namely we prove that any abelian variety that is not affine has a nontrivial strongly abelian subvariety.

In the second paper in the sequence, [5], we accomplish Goal 2 in the following sense: we reduce the classification of minimal affine varieties to the classification of simple rings. Each minimal affine variety has an associated simple ring, and each simple ring is associated to some minimal affine varieties. This is a many-to-one correspondence between minimal affine varieties and simple rings. We completely explain the relationship between the varieties and the rings.

In the third paper in the sequence, [6], we make partial progress on Goal 3. Namely, we classify those minimal strongly abelian varieties that have a finite bound on the essential arities of their terms. Here, when we say 'we classify', we mean that we reduce the classification of these varieties to the classification of simple monoids with zero. Also in the third paper of the sequence we show that there are minimal strongly abelian varieties that do not have a finite bound on the essential arities of their terms, thereby showing that there is more work to do to complete the classification of minimal strongly abelian varieties.

2. Terminology and notation

An algebraic language $\mathcal{L}$ is determined by a function $\alpha : F \to \omega$ where F is a set of operation symbols and α assigns arity. An algebra for this language is a pair $\langle A; F\rangle$ where A is a nonempty set and for each symbol $f \in F$ there is a fixed interpretation $f^{\mathbf{A}} : A^{\alpha(f)} \to A$ of that symbol as an $\alpha(f)$-ary operation on A.

Let $X = \{x_1, \ldots\}$ be a set of variables. The set $\mathcal{T}$ of all terms in X in a language $\mathcal{L}$ is defined recursively by stipulating that (i) $X \subseteq \mathcal{T}$, and (ii) if $f \in F$, $\alpha(f) = k$, and $t_1, \ldots, t_k \in \mathcal{T}$, then $f(t_1, \ldots, t_k) \in \mathcal{T}$. The assignment $f \mapsto f^{\mathbf{A}}$, which assigns an operation table to a symbol, can be extended to terms $t \mapsto t^{\mathbf{A}}$. We call the interpretation $t^{\mathbf{A}}$ of t the term operation of $\mathbf{A}$ associated to the term t.

An identity in $\mathcal{L}$ is a pair of terms, written $s \approx t$. The identity $s \approx t$ is satisfied by $\mathbf{A}$, written $\mathbf{A} \models s \approx t$, if $s^{\mathbf{A}} = t^{\mathbf{A}}$. Given a set Σ of identities, the class $\mathcal{V}$ of all $\mathcal{L}$-algebras satisfying Σ is called the variety axiomatized by Σ.

We shall use Birkhoff's Theorem, which asserts that the smallest variety containing a class $\mathcal{K}$ of $\mathcal{L}$-algebras is the class $\mathsf{H}\,\mathsf{S}\,\mathsf{P}(\mathcal{K})$ of homomorphic images of subalgebras of products of algebras in $\mathcal{K}$.

A subvariety of a variety $\mathcal{V}$ is a subclass of $\mathcal{V}$ that is a variety. A variety is trivial if it consists of 1-element algebras only. A variety is minimal if it is not trivial, but any proper subvariety is trivial.

The full constant expansion of $\mathbf{A}$ is the algebra $\mathbf{A}_A = \langle A; F \cup \{c_a \mid a \in A\}\rangle$ obtained from $\mathbf{A}$ by adding a new 0-ary (constant) symbol c_a for each element $a \in A$. By a polynomial operation of $\mathbf{A}$ we mean a term operation of $\mathbf{A}_A$.

A $1,1$-matrix of $\mathbf{A}$ is a 2×2 matrix of elements of A of the form

$$\begin{bmatrix} t(\mathbf{a},\mathbf{u}) & t(\mathbf{a},\mathbf{v}) \\ t(\mathbf{b},\mathbf{u}) & t(\mathbf{b},\mathbf{v}) \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in A^{2\times 2} \tag{2.1}$$

where $t(\mathbf{x},\mathbf{y})$ is a polynomial of $\mathbf{A}$ and $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v}$ are tuples of elements of A. The set of $1,1$-matrices is invariant under the operations of swapping rows, swapping columns, and matrix transpose.

A has property "X", or "is X", if the corresponding implications hold for all $1,1$-matrices in (2.1):

- (X = abelian) provided $p = q$ implies $r = s$. (Equivalently, if $p = r$ implies $q = s$.)
- (X = rectangular) provided that, for some compatible partial order on $\mathbf{A}$, $\geq$, it is the case that $u \geq q$ and $u \geq r$ together imply $u \geq s$.
- (X = strongly rectangular) provided $q = r$ implies $r = s$.
- (X = strongly abelian) (same as abelian + strongly rectangular).
- (X = affine) (same as abelian + has a Maltsev operation).

A variety "is X" if all of its algebras are.

All of these concepts will be used in this paper. What we have just written is not enough to understand what follows, so please see Chapters 2 and 5 of [3] for more detail about these concepts when necessary.

3. Abelian and affine algebras

Our main goal in this section is to prove that if $\mathcal{V}$ is an abelian variety that is not an affine variety, then $\mathcal{V}$ contains a nontrivial strongly abelian subvariety. Applying this to the situation where $\mathcal{V}$ is a minimal variety, we obtain that any minimal abelian variety is affine or strongly abelian.

The path we follow in this section is to prove the following sequentially stronger Facts about an arbitrary abelian variety $\mathcal{V}$ that is not affine:

(I) $\mathcal{V}$ contains an algebra with a nontrivial strongly abelian congruence.
(II) $\mathcal{V}$ contains a nontrivial strongly abelian algebra.
(III) $\mathcal{V}$ contains a nontrivial strongly abelian subvariety.

The following theorem establishes Fact (I).

Theorem 3.1. *Let $\mathcal{V}$ be an abelian variety. If $\mathcal{V}$ is not affine, then there is an algebra $\mathbf{A} \in \mathcal{V}$ that has a nontrivial strongly abelian congruence.*

Proof. We prove the contrapositive of the second sentence in the theorem statement. Namely, under the hypothesis that $\mathcal{V}$ is abelian, we show that if $\mathcal{V}$ contains no algebra $\mathbf{A}$ with a nontrivial strongly abelian congruence, then $\mathcal{V}$ is affine.

If $\mathcal{V}$ has no algebra $\mathbf{A}$ with a nontrivial strongly abelian congruence, then Theorem 3.13 of [3] proves that $\mathcal{V}$ satisfies a nontrivial idempotent Maltsev condition. By Theorem 3.21 of [3], any variety that satisfies a nontrivial idempotent Maltsev condition has a *join term*, which is a term whose associated term operation acts as a semilattice join operation on the blocks of any rectangular tolerance relation of any algebra in the variety. But no subset of more than one element in an abelian algebra can be closed under a semilattice term operation, because this would realize a nontrivial semilattice as a subalgebra of a reduct of an abelian algebra. Subalgebras of reducts of abelian algebras are abelian, and no semilattice of more than one element is abelian. This shows that blocks of rectangular tolerances in $\mathcal{V}$ are singleton sets, which is another way of saying that $\mathcal{V}$ contains no algebra with a nontrivial rectangular tolerance.

By Theorem 5.25 of [3], the fact that $\mathcal{V}$ omits nontrivial rectangular tolerances is equivalent to the fact that $\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in the variety of semilattices. Finally, Theorem 4.10 of [9], proves that if $\mathcal{V}$ is any variety satisfying an idempotent Maltsev condition which fails in the variety of semilattices, then abelian algebras in $\mathcal{V}$ are affine. Altogether, this shows that if $\mathcal{V}$ is abelian and no algebra in $\mathcal{V}$ has a nontrivial strongly abelian congruence, then $\mathcal{V}$ is affine. □

This concludes the proof of Fact (I). Our next goal is to prove Fact (II): if $\mathcal{V}$ is abelian but not affine, then $\mathcal{V}$ contains a nontrivial strongly abelian algebra.

The following notation will be needed for Lemma 3.2, which is a result proved in [4] (Lemma 2.1 of that paper). Assume that $\mathbf{A}$ is abelian and $\theta \in \mathrm{Con}(\mathbf{A})$ is strongly abelian. Let $\mathbf{A}(\theta)$ be the subalgebra of $\mathbf{A} \times \mathbf{A}$ supported by the graph of θ. Let Δ be the congruence on $\mathbf{A}(\theta)$ generated by $D \times D$ where $D = \{(a, a) \mid a \in A\}$ is the diagonal. D is a Δ-class, because $\mathbf{A}$ is abelian. Let $\mathbf{S} = \mathbf{S}_{\mathbf{A},\theta} := \mathbf{A}(\theta)/\Delta$. Let $0 = D/\Delta \in S$.

Lemma 3.2. *Let $\mathcal{V}$ be an abelian variety, and suppose that θ is a nontrivial strongly abelian congruence on some $\mathbf{A} \in \mathcal{V}$. Let $\mathbf{S} = \mathbf{S}_{\mathbf{A},\theta}$ and let $0 = D/\Delta \in S$. The following are true:*

(1) $\mathbf{S}$ *has more than one element.*
(2) $\{0\}$ *is a 1-element subuniverse of* $\mathbf{S}$.
(3) $\mathbf{S}$ *has "Property P": for every n-ary polynomial $p(\mathbf{x})$ of $\mathbf{S}$ and every tuple $\mathbf{s} \in S^n$*

$$p(\mathbf{s}) = 0 \quad \textit{implies} \quad p(\mathbf{0}) = 0,$$

where $\mathbf{0} = (0, 0, \ldots, 0)$.
(4) *Whenever $t(x_1, \ldots, x_n)$ is a $\mathcal{V}$-term and*

$$\mathcal{V} \models t(\mathbf{x}) \approx t(\mathbf{y})$$

where $\mathbf{x}$ and $\mathbf{y}$ are tuples of not necessarily distinct variables which differ in the ith position, then the term operation $t^{\mathbf{S}}(x_1, \ldots, x_n)$ is independent of its ith variable.
(5) $\mathbf{S}$ *has a congruence σ such that the algebra $\mathbf{S}/\sigma$ satisfies (1)–(4) of this lemma, and $\mathbf{S}/\sigma$ also has a compatible partial order $\leq$ such that $0 \leq s$ for every $s \in (S/\sigma)$.*

□

This lemma puts us in position to establish Fact (II):

Theorem 3.3. *If $\mathcal{V}$ is an abelian variety that is not affine, then $\mathcal{V}$ contains a nontrivial strongly abelian algebra.*

Proof. By Theorem 3.1, the assumption that $\mathcal{V}$ is abelian but nonaffine guarantees that there is some $\mathbf{A} \in \mathcal{V}$ that has some nontrivial strongly abelian congruence. By Lemma 3.2, these data can be used to construct a nontrivial algebra $\mathbf{T} := \mathbf{S}/\sigma \in \mathcal{V}$ that has a compatible partial order $\leq$ and a singleton subuniverse $\{0\}$ such that 0 is the least element under the partial order. We proceed from this point.

Observe that if $a, b \in T$ satisfy $a \geq b$, then from $a \geq b \geq 0$ we derive that $f(a) \geq f(b) \geq f(0)$ (≥ 0) for any unary polynomial $f \in \mathrm{Pol}_1(\mathbf{T})$. In particular, if

$f(a) = 0$ we must also have $f(b) = 0$. Let us define a coarser quasiorder $\sqsupseteq$ on $\mathbf{T}$ by this rule: for $a, b \in T$, let $a \sqsupseteq b$ if

$$f(a) = 0 \Rightarrow f(b) = 0 \tag{3.1}$$

holds for all $f \in \mathrm{Pol}_1(\mathbf{T})$. The relation $\sqsupseteq$ is reflexive, transitive, and compatible with unary polynomials, so it is compatible with all polynomials. Therefore $\sqsupseteq$ is a compatible quasiorder on $\mathbf{T}$. From the first two sentences of this paragraph we see that $\sqsupseteq$ extends $\geq$ (i.e., $\sqsupseteq$ is a coarsening of $\geq$). This is enough to imply that 0 is a least element with respect to $\sqsupseteq$.

Let $\theta = \sqsupseteq \cap \sqsupseteq^{\cup}$. By considering what happens in (3.1) when $0 \sqsupseteq b$ and $f(x) = x$ one sees that $0 \sqsupseteq b$ implies $b = 0$. Hence $0/\theta = \{0\}$, from which it follows that θ is a proper congruence of $\mathbf{T}$.

We let $\mathbf{T}' = \mathbf{T}/\theta$, $0' = 0/\theta$, and $\geq' = \sqsupseteq/\theta$. Now $\mathbf{T}'$ is nontrivial, has a compatible partial order $\geq'$ with least element $0'$, and with respect to this partial order we have $a \geq' b$ if and only if

$$f(a) = 0' \Rightarrow f(b) = 0'$$

for all unary polynomials $f \in \mathrm{Pol}_1(\mathbf{T}')$.

Our new algebra $\mathbf{T}'$ is a quotient of the original algebra, has all properties attributed to $\mathbf{T}$ in the first paragraph of this proof, but now we have strengthened the implication "$a \geq' b$ in $\mathbf{T}'$ implies $f(a) = 0' \Rightarrow f(b) = 0'$ for all unary polynomials $f \in \mathrm{Pol}_1(\mathbf{T}')$" to a bi-implication. We now replace $\mathbf{T}$ with $\mathbf{T}'$, drop all primes, and assume that

$$a \geq b \text{ in } \mathbf{T} \text{ if and only if } f(a) = 0 \Rightarrow f(b) = 0. \tag{3.2}$$

It should be pointed out that the reflexivity, transitivity, and compatibility of $\geq$ implies that $\mathbf{T}$ satisfies Property P of Lemma 3.2. For suppose that $\mathbf{s} = (s_1, \ldots, s_n)$ and $p(\mathbf{s}) = 0$ for some polynomial p of $\mathbf{T}$. Since $s_i \geq 0$ for all i we derive from (3.2) that

$$\begin{aligned} p(\mathbf{s}) = p(s_1, s_2, s_3, \ldots, s_n) = 0 \quad &\Rightarrow p(0, s_2, s_3, \ldots, s_n) = 0 \\ &\Rightarrow p(0, 0, s_3, \ldots, s_n) = 0 \\ &\vdots \\ &\Rightarrow p(0, 0, 0, \ldots, 0) = 0, \end{aligned}$$

which is the assertion of Property P.

Claim 3.4. *The total binary relation* $1 \in \mathrm{Con}(\mathbf{T})$ *rectangulates itself with respect to* $\geq$.

Proof of Claim. Assume that

$$\begin{bmatrix} t(\mathbf{a},\mathbf{u}) & t(\mathbf{a},\mathbf{v}) \\ t(\mathbf{b},\mathbf{u}) & t(\mathbf{b},\mathbf{v}) \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \tag{3.3}$$

is a $1,1$-matrix and that $u \geq q, r$. Our goal is to prove that $u \geq s$. If $u \not\geq s$, then according to (3.2) there is a unary polynomial f such that $f(u) = 0$ and $f(s) \neq 0$. Since $u \geq q, r$ and since 0 is the least element under $\geq$, we get that $0 = f(u) \geq f(q), f(r) \geq f(0)$, so in fact $0 = f(u) = f(q) = f(r) = f(0)$. Prefixing the polynomial t in the left matrix of (3.3) with the polynomial f, we obtain a $1,1$-matrix of the form

$$\begin{bmatrix} ft(\mathbf{a},\mathbf{u}) & ft(\mathbf{a},\mathbf{v}) \\ ft(\mathbf{b},\mathbf{u}) & ft(\mathbf{b},\mathbf{v}) \end{bmatrix} = \begin{bmatrix} f(p) & f(q) \\ f(r) & f(s) \end{bmatrix} = \begin{bmatrix} p' & 0 \\ 0 & s' \end{bmatrix} \tag{3.4}$$

with $s' \neq 0$. That is,

$$ft(\mathbf{a},\mathbf{v}) = 0 = ft(\mathbf{b},\mathbf{u}), \tag{3.5}$$

while $ft(\mathbf{b},\mathbf{v}) \neq 0$. Employing Property P in the forms $ft(\underline{\mathbf{a}},\mathbf{v}) = 0 \Rightarrow ft(\underline{\mathbf{0}},\mathbf{v}) = 0$ and $ft(\underline{\mathbf{b}},\mathbf{u}) = 0 \Rightarrow ft(\underline{\mathbf{0}},\mathbf{u}) = 0$, we get from (3.5) that

$$ft(\underline{\mathbf{0}},\mathbf{v}) = 0 = ft(\underline{\mathbf{0}},\mathbf{u}). \tag{3.6}$$

Since $\mathbf{T}$ is abelian, we derive from (3.6) that

$$ft(\underline{\mathbf{b}},\mathbf{v}) = ft(\underline{\mathbf{b}},\mathbf{u}). \tag{3.7}$$

The left side of (3.7) equals s' while the right side equals 0, yielding $s' = 0$, contrary to the fact stated after the line containing (3.4). The claim is proved. ∎

Claim 3.5. *The total binary relation* $1 \in \mathrm{Con}(\mathbf{T})$ *strongly rectangulates itself.*

Proof of Claim. Assume that

$$\begin{bmatrix} t(\mathbf{a},\mathbf{u}) & t(\mathbf{a},\mathbf{v}) \\ t(\mathbf{b},\mathbf{u}) & t(\mathbf{b},\mathbf{v}) \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \tag{3.8}$$

is a $1,1$-matrix and that $q = r$. Our goal is to prove that $r = s$. Note that by taking $u = q = r$ we get from rectangulation with respect to $\geq$ (Claim 3.4) that, since $u \geq q, r$, we must have $u \geq s$, so in particular we have $r \geq s$. If we do not have $r = s$ as desired, then we must have $s \not\geq r$. In this case there is a unary polynomial f such that $f(s) = 0$ and $f(r) \neq 0$. By prefixing the polynomial t in the left matrix in (3.8) by f, we obtain a matrix of the form

$$\begin{bmatrix} ft(\mathbf{a},\mathbf{u}) & ft(\mathbf{a},\mathbf{v}) \\ ft(\mathbf{b},\mathbf{u}) & ft(\mathbf{b},\mathbf{v}) \end{bmatrix} = \begin{bmatrix} f(p) & f(q) \\ f(q) & f(s) \end{bmatrix} = \begin{bmatrix} p' & q' \\ q' & 0 \end{bmatrix}$$

with $q' = f(q) = f(r) \neq 0$. We also have (by rectangulation with respect to $\geq$) that any u that majorizes the cross diagonal in

$$\begin{bmatrix} p' & q' \\ q' & 0 \end{bmatrix},$$

like $u = q'$, also majorizes the main diagonal, i.e. $q' \geq p'$. Similarly, any u that majorizes the main diagonal, like $u = p'$, also majorizes the cross diagonal, i.e. $p' \geq q'$. In particular, $p' = q'$ and we have

$$\begin{bmatrix} ft(\mathbf{a},\mathbf{u}) & ft(\mathbf{a},\mathbf{v}) \\ ft(\mathbf{b},\mathbf{u}) & ft(\mathbf{b},\mathbf{v}) \end{bmatrix} = \begin{bmatrix} q' & q' \\ q' & 0 \end{bmatrix}.$$

This is a failure of the term condition (which defines abelianness), thereby proving the claim. ▮

We complete the proof of Theorem 3.3 by noting that an algebra is strongly abelian if and only if it is abelian and strongly rectangular. Since we have shown that $\mathbf{T}$ is strongly rectangular, and we selected $\mathbf{T}$ from an abelian variety, we conclude that $\mathbf{T}$ is strongly abelian. □

Next on our agenda is to prove Fact (III), which asserts that, if $\mathcal{V}$ is abelian but not affine, then $\mathcal{V}$ has a nontrivial subvariety that is strongly abelian. We shall require the following lemma, which is an extension of [10, Theorem 7.1].

Recall the class operators H, S, and P we introduced briefly in Section 2. Namely, for a class $\mathcal{K}$ of similar algebras $\mathsf{H}(\mathcal{K})$ denotes the class of algebras that are homomorphic images of members of $\mathcal{K}$, $\mathsf{S}(\mathcal{K})$ denotes the class of algebras isomorphic to subalgebras of members of $\mathcal{K}$, and $\mathsf{P}(\mathcal{K})$ denotes the class of algebras isomorphic to products of members of $\mathcal{K}$. Each of the classes $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$, and $\mathsf{P}(\mathcal{K})$ has been defined so that it is closed under isomorphism.

Lemma 3.6. *Assume that every finitely generated subalgebra of* $\mathbf{A}$ *is strongly solvable. If* $\mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$ *consists of abelian algebras, then* $\mathsf{H}\,\mathsf{S}(\mathbf{A})$ *consists of strongly abelian algebras.*

Proof. For the first step of the proof we invoke [10, Theorem 7.1], which proves the following:

Claim 3.7. *Assume that every finitely generated subalgebra of* $\mathbf{A}$ *is strongly solvable. If* $\mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$ *consists of abelian algebras, then* $\mathbf{A}$ *is strongly abelian.*

The conclusion that $\mathbf{A}$ is strongly abelian in Claim 3.7 implies that every algebra in $\mathsf{S}(\mathbf{A})$ is also strongly abelian, since the strong abelian property is expressible by universal sentences. However it is not an immediate consequence of Claim 3.7 that the class $\mathsf{H}\,\mathsf{S}(\mathbf{A})$ consists of strongly abelian algebras. For this we must show that if $\mathbf{B} \in \mathsf{S}(\mathbf{A})$ and $\theta \in \mathrm{Con}(\mathbf{B})$, then $\mathbf{B}/\theta$ is also strongly abelian.

Choose and fix $\mathbf{B} \in \mathsf{S}(\mathbf{A})$ and $\theta \in \mathrm{Con}(\mathbf{B})$. Recall (from Section 2) that an algebra is strongly abelian if and only if it is both abelian and strongly rectangular. We are assuming that all algebras in $\mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$ are abelian, and $\mathbf{B}/\theta$ is in $\mathsf{H}\,\mathsf{S}(\mathbf{A})$, which is a subclass of the abelian class $\mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$, so to prove the lemma it suffices to prove that $\mathbf{B}/\theta$ is strongly rectangular.

This will be a proof by contradiction. Our aim will be to obtain a contradiction from the assumptions that: $\mathbf{A}$ is strongly abelian (which we get from Claim 3.7), $\mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$ consists of abelian algebras, $\mathbf{B} \leq \mathbf{A}$, $\theta \in \mathrm{Con}(\mathbf{B})$, but $\mathbf{B}/\theta$ is not strongly rectangular. Observe that the assumptions that $\mathbf{A}$ is strongly abelian and $\mathbf{B} \leq \mathbf{A}$ imply that $\mathbf{B}$ is strongly abelian. We reiterate the observation of the last paragraph that the assumption that $\mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$ consists of abelian algebras implies that $\mathbf{B}/\theta$ is abelian.

The assumption that $\mathbf{B}/\theta$ is not strongly rectangular means exactly that there is a $1,1$-matrix in $\mathbf{B}$ of the form

$$\begin{bmatrix} t(\mathbf{a},\mathbf{u}) & t(\mathbf{a},\mathbf{v}) \\ t(\mathbf{b},\mathbf{u}) & t(\mathbf{b},\mathbf{v}) \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \tag{3.9}$$

with $q \equiv_\theta r$ but $r \not\equiv_\theta s$.

Claim 3.8. *No two elements of the second matrix of (3.9) which come from the same row or column are* θ*-related.*

Proof of Claim. We explain why p and r, the two elements of the first column of the second matrix in (3.9), are not θ-related and omit the proofs of the other (similar) cases.

Assume that $p \equiv_\theta r$. Since $\mathbf{B}/\theta$ is abelian, the ordinary term condition holds in $\mathbf{B}/\theta$, i.e., $C(1,1;\theta)$ holds in $\mathbf{B}$. From

$$t(\mathbf{a},\underline{\mathbf{u}}) = p \equiv_\theta r = t(\mathbf{b},\underline{\mathbf{u}})$$

we derive

$$t(\mathbf{a},\underline{\mathbf{v}}) = q \equiv_\theta s = t(\mathbf{b},\underline{\mathbf{v}})$$

by replacing the underlined $\mathbf{u}$'s with $\mathbf{v}$'s. In short, if the two elements of the first column of the second matrix in (3.9) are θ-related, then the two elements in the parallel column must also be θ-related. This, together with the earlier condition $q \equiv_\theta r$, which asserts that the elements along the cross diagonal are θ-related, yields that all elements of the matrix are θ-related. (That is, $p \equiv_\theta r \equiv_\theta q \equiv_\theta s$). This is in contradiction to the condition $r \not\equiv_\theta s$. What we have contradicted was the assumption that the elements p and r, which come from the same column of the second matrix in (3.9), are θ-related. ∎

Next, let $D = \{(z, z) \mid z \in B\}$ be the diagonal subuniverse of $\mathbf{B}^2$. Let $\mathbf{C} \leq \mathbf{B}^2$ be the subalgebra of $\mathbf{B}^2$ generated by D and all pairs (u_i, v_i), $i = 1, \ldots, n$, where $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in the first matrix from (3.9). Observe that the pairs (p, q) and (r, s) both belong to $\mathbf{C}$, since

$$(p, q) = (t(\mathbf{a}, \mathbf{u}), t(\mathbf{a}, \mathbf{v})) = t(\underbrace{(a_1, a_1), (a_2, a_2), \ldots, (u_1, v_1), (u_2, v_2), \ldots}_{\text{generators of } \mathbf{C}}) \in C$$

and

$$(r, s) = (t(\mathbf{b}, \mathbf{u}), t(\mathbf{b}, \mathbf{v})) = t(\underbrace{(b_1, b_1), (b_2, b_2), \ldots, (u_1, v_1), (u_2, v_2), \ldots}_{\text{generators of } \mathbf{C}}) \in C.$$

Let γ be the principal congruence of $\mathbf{C}$ generated by the single pair (of pairs) $\big((p, q), (r, s)\big)$.

Claim 3.9. *The diagonal $D \subseteq C$ is a union of γ-classes. Moreover, if $\big((c, c), (d, d)\big) \in \gamma$, then $(c, d) \in \theta$.*

Proof of Claim. Since γ is the congruence generated by the pair $\big((p, q), (r, s)\big)$, it follows from Maltsev's Congruence Generation Lemma that to prove this claim it suffices to establish that if f is any unary polynomial of $\mathbf{C}$, and $f\big((p, q)\big) = (c, c) \in D$, then $f\big((r, s)\big) = (d, d) \in D$ for some d satisfying $(c, d) \in \theta$.

Assume that f is a unary polynomial of $\mathbf{C}$ and that $f\big((p, q)\big) = (c, c)$ for some c. Unary polynomials of $\mathbf{C}$ have the form

$$f\big((x, y)\big) = g\big((x, y), (\mathbf{u}, \mathbf{v})\big) = \big(g(x, \mathbf{u}), g(y, \mathbf{v})\big)$$

where $g(x, \mathbf{z})$ is a polynomial of $\mathbf{B}$, since $\mathbf{C}$ is generated by D and all pairs (u_i, v_i), $i = 1, \ldots, n$, where $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. Thus $f\big((p, q)\big) = (c, c)$ can be rewritten

$$g(p, \mathbf{u}) = c = g(q, \mathbf{v}) \tag{3.10}$$

for some polynomial g of $\mathbf{B}$. More fully, this is

$$g(t(\mathbf{a}, \mathbf{u}), \mathbf{u}) = c = g(t(\mathbf{a}, \mathbf{v}), \mathbf{v}).$$

Apply the term condition in the abelian algebra $\mathbf{B}$ to change the underlined $\mathbf{a}$'s to $\mathbf{b}$'s below, so from

$$g(t(\underline{\mathbf{a}}, \mathbf{u}), \mathbf{u}) = c = g(t(\underline{\mathbf{a}}, \mathbf{v}), \mathbf{v})$$

we get

$$g(t(\underline{\mathbf{b}}, \mathbf{u}), \mathbf{u}) = g(t(\underline{\mathbf{b}}, \mathbf{v}), \mathbf{v}).$$

Less fully, this equality may be rewritten

$$g(r, \mathbf{u}) = d = g(s, \mathbf{v}) \tag{3.11}$$

for some d. In other words, the fact that $\mathbf{B}$ is abelian implies that if $f\big((p, q)\big) = (c, c) \in D$ for some c, then there is some d such that $f\big((r, s)\big) = \big(g(r, \mathbf{u}), g(s, \mathbf{v})\big) = (d, d)$.

It remains to argue that we must have $(c, d) \in \theta$. Consider the following $1,1$-matrix of $\mathbf{B}$, where two of the entries in the left matrix can be determined from Equation (3.10):

$$\begin{bmatrix} g(p, \mathbf{u}) & g(p, \mathbf{v}) \\ g(q, \mathbf{u}) & g(q, \mathbf{v}) \end{bmatrix} = \begin{bmatrix} c & * \\ * & c \end{bmatrix}.$$

The off-diagonal entries can be determined from the diagonal entries, since $\mathbf{B}$ is strongly abelian. Namely, all four entries must equal c, yielding $g(p, \mathbf{u}) = g(p, \mathbf{v}) = g(q, \mathbf{u}) = g(q, \mathbf{v}) = c$. In particular, this and Equation (3.11) yield $(g(q, \mathbf{u}), g(r, \mathbf{u})) = (c, d)$. This shows that the pair (c, d) is a polynomial translate of the pair (q, r) via the translation $x \mapsto g(x, \mathbf{u})$. Since $(q, r) \in \theta \in \mathrm{Con}(\mathbf{B})$, according to the line after (3.9), and $g(x, \mathbf{u})$ is a unary polynomial of $\mathbf{B}$ it follows that $(c, d) \in \theta$. ▮

Claim 3.10. *$\mathbf{C}/\gamma$ is not abelian.*

Proof of Claim. We argue that $C(1, 1; \gamma)$ fails in $\mathbf{C}$. For this it suffices to write down a bad $1,1$-matrix of $\mathbf{C}$:

$$\begin{bmatrix} t\big((\mathbf{a}, \mathbf{a}), \underline{(\mathbf{u}, \mathbf{v})}\big) & t\big((\mathbf{b}, \mathbf{b}), \underline{(\mathbf{u}, \mathbf{v})}\big) \\ t\big((\mathbf{a}, \mathbf{a}), \underline{(\mathbf{u}, \mathbf{u})}\big) & t\big((\mathbf{b}, \mathbf{b}), \underline{(\mathbf{u}, \mathbf{u})}\big) \end{bmatrix} = \begin{bmatrix} (p, q) & (r, s) \\ (p, p) & (r, r) \end{bmatrix}. \tag{3.12}$$

(One should check that this <u>is</u> a $1,1$-matrix in $\mathbf{C}$, i.e., that t is being applied to elements of $\mathbf{C}$: $(a_i, a_i), (b_j, b_j), (u_k, u_k), (u_\ell, v_\ell) \in C$.)

Our goal is to show that the first matrix in (3.12) expresses a failure of $C(1, 1; \gamma)$, which in more standard notation might be written:

$$t\big((\mathbf{a}, \mathbf{a}), \underline{(\mathbf{u}, \mathbf{v})}\big) \equiv_\gamma t\big((\mathbf{b}, \mathbf{b}), \underline{(\mathbf{u}, \mathbf{v})}\big),$$

while changing the underlined $(\mathbf{u}, \mathbf{v})$'s to $(\mathbf{u}, \mathbf{u})$'s produces

$$t\big((\mathbf{a}, \mathbf{a}), \underline{(\mathbf{u}, \mathbf{u})}\big) \not\equiv_\gamma t\big((\mathbf{b}, \mathbf{b}), \underline{(\mathbf{u}, \mathbf{u})}\big).$$

To see that this truly is a failure of $C(1, 1; \gamma)$, notice that the elements on the first row of the second matrix in (3.12) are indeed γ-related, since γ was defined to be the congruence of $\mathbf{C}$ generated by $\big((p, q), (r, s)\big)$. But notice also that the elements on the second row of the second matrix in (3.12) cannot possibly be γ-related. For, we proved in Claim 3.9 that γ-related pairs of the form $\big((c, c), (d, d)\big)$ must satisfy $(c, d) \in \theta$, and we proved that $(p, r) \notin \theta$ in Claim 3.8. Hence $\big((p, p), (r, r)\big) \notin \gamma$. ▮

To summarize, in the main portion of the proof we showed that when $\mathbf{B}$ is strongly abelian, $\theta \in \mathrm{Con}(\mathbf{B})$, and $\mathbf{B}/\theta$ is abelian but not strongly rectangular, then there is an algebra $\mathbf{C}/\gamma \in \mathsf{H}\,\mathsf{S}(\mathbf{B}^2)$ that is not abelian. It follows that when $\mathbf{A}$ is strongly abelian, $\mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$ consists of abelian algebras, $\mathbf{B} \leq \mathbf{A}$, $\theta \in \mathrm{Con}(\mathbf{B})$, and $\mathbf{B}/\theta$ is not strongly rectangular, then there is an algebra $\mathbf{C}/\gamma \in \mathsf{H}\,\mathsf{S}(\mathbf{B}^2) \subseteq \mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$ that is not abelian. This was what we needed to establish, as one can verify by consulting the third paragraph following Claim 3.7. □

Theorem 3.11. *Let $\mathcal{V}$ be an abelian variety. If $\mathcal{K}$ is any subclass of $\mathcal{V}$ that consists of strongly abelian algebras, then the subvariety of $\mathcal{V}$ generated by $\mathcal{K}$ is strongly abelian. (Equivalently, the subclass of all strongly abelian algebras in $\mathcal{V}$ is a subvariety of $\mathcal{V}$.)*

Proof. Since $\mathcal{K}$ consists of strongly abelian algebras, and the property of being strongly abelian is expressible by a family of universal Horn sentences, the class $\mathsf{S}\,\mathsf{P}(\mathcal{K})$ of algebras isomorphic to subalgebras of products of members of $\mathcal{K}$ consists of strongly abelian algebras. Any $\mathbf{A} \in \mathsf{S}\,\mathsf{P}(\mathcal{K})$ will be strongly abelian, hence will have the property that its finitely generated subalgebras are strongly abelian. Moreover, since $\mathbf{A} \in \mathcal{V}$, and $\mathcal{V}$ is abelian, the class $\mathsf{H}\,\mathsf{S}(\mathbf{A}^2)$ will consist of abelian algebras. By Lemma 3.6, the class $\mathsf{H}\,\mathsf{S}(\mathbf{A})$ consists of strongly abelian algebras. Since $\mathbf{A} \in \mathsf{S}\,\mathsf{P}(\mathcal{K})$ was arbitrary, this means that $\mathsf{H}\,\mathsf{S}(\mathsf{S}\,\mathsf{P}(\mathcal{K})) = \mathsf{H}\,\mathsf{S}\,\mathsf{P}(\mathcal{K})$ consists of strongly abelian algebras. Since $\mathsf{H}\,\mathsf{S}\,\mathsf{P}(\mathcal{K})$ is the variety generated by $\mathcal{K}$, the theorem is proved. □

Remark 3.12. For *locally finite* varieties, the result stated in Theorem 3.11 is due to Matt Valeriote. It is known from Tame Congruence Theory that if $\mathcal{V}$ is a locally finite variety, then the subclass $\mathcal{S}$ of locally strongly solvable algebras in $\mathcal{V}$ is a subvariety of $\mathcal{V}$. Valeriote proved in [14] that the following are equivalent for any locally finite *abelian* variety $\mathcal{S}$:

(1) $\mathcal{S}$ is locally strongly solvable.
(2) $\mathcal{S}$ is strongly abelian.

Here is how you deduce Theorem 3.11 in the locally finite case from the statements just made. Assume that $\mathcal{V}$ is a locally finite abelian variety. Let $\mathcal{S} \subseteq \mathcal{V}$ be the subvariety of $\mathcal{V}$ consisting of locally strongly solvable algebras in $\mathcal{V}$. Let $\mathcal{K} \subseteq \mathcal{V}$ be the subclass of strongly abelian algebras of $\mathcal{V}$. Since a strongly abelian algebra is locally strongly solvable, $\mathcal{K} \subseteq \mathcal{S}$. Valeriote's Theorem applied to the subvariety $\mathcal{S}$ shows that $\mathcal{S} \subseteq \mathcal{K}$, hence $\mathcal{K} = \mathcal{S}$, which shows that $\mathcal{K}$ is a subvariety of $\mathcal{V}$.

Theorem 3.13. *If $\mathcal{V}$ is an abelian variety that is not affine, then $\mathcal{V}$ contains a nontrivial strongly abelian subvariety.*

Proof. Theorem 3.3 shows that if $\mathcal{V}$ is abelian but not affine, then there is some nontrivial algebra in $\mathbf{A} \in \mathcal{V}$ that is strongly abelian. By Theorem 3.11, the subvariety of $\mathcal{V}$ generated by $\mathbf{A}$ is a nontrivial strongly abelian subvariety of $\mathcal{V}$. □

Corollary 3.14. *Any minimal abelian variety of algebras is affine or strongly abelian.* □

References

[1] Keith A. Kearnes, *Idempotent simple algebras.* in Logic and Algebra, Lecture Notes in Pure and Appl. Math. 180, 1996, 529–572.
[2] Keith A. Kearnes, *Almost all minimal idempotent varieties are congruence modular.* Algebra Universalis **44** (2000), 39–45.
[3] Keith A. Kearnes and Emil W. Kiss, *The shape of congruence lattices.* Mem. Amer. Math. Soc. **222** (2013), no. 1046.
[4] Keith A. Kearnes, Emil W. Kiss, and Ágnes Szendrei, *Varieties whose finitely generated members are free.* Algebra Universalis **79** (2018), no. 1, 79:3.
[5] Keith A. Kearnes, Emil W. Kiss, and Ágnes Szendrei, *Minimal abelian varieties of algebras, II*, manuscript.
[6] Keith A. Kearnes, Emil W. Kiss, and Ágnes Szendrei, *Minimal abelian varieties of algebras, III*, manuscript.
[7] Keith A. Kearnes, Emil W. Kiss, and Matthew A. Valeriote, *Minimal sets and varieties.* Trans. Amer. Math. Soc. **350** (1998), no. 1, 1–41.
[8] Keith A. Kearnes and Ágnes Szendrei, *A characterization of minimal locally finite varieties.* Trans. Amer. Math. Soc. **349** (1997), no. 5, 1749–1768.
[9] Keith A. Kearnes and Ágnes Szendrei, *The relationship between two commutators.* Internat. J. Algebra Comput. **8** (1998), 497–531.
[10] Ralph McKenzie and Matthew A. Valeriote, *The structure of decidable locally finite varieties.* Progress in Mathematics, **79**. Birkhäuser Boston, Inc., Boston, MA, 1989.
[11] Ágnes Szendrei, *A survey on strictly simple algebras and minimal varieties.* Universal algebra and quasigroup theory (Jadwisin, 1989), 209–239, Res. Exp. Math., **19**, Heldermann, Berlin, 1992.

[12] Ágnes Szendrei, *Strongly abelian minimal varieties.* Acta Sci. Math. (Szeged) **59** (1994), no. 1-2, 25–42.
[13] Ágnes Szendrei, *Maximal non-affine reducts of simple affine algebras.* Algebra Universalis **34** (1995), no. 1, 144–174.
[14] Matthew Anthony Valeriote, On decidable locally finite varieties (universal algebra, abelian, multi-unary), PhD. dissertation, University of California, Berkeley. 1986. 139 pp.

(Keith A. Kearnes) Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, USA
E-mail address: kearnes@colorado.edu

(Emil W. Kiss) Loránd Eötvös University, Department of Algebra and Number Theory, H–1117 Budapest, Pázmány Péter sétány 1/c., Hungary
E-mail address: ewkiss@cs.elte.hu

(Ágnes Szendrei) Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, USA
E-mail address: szendrei@colorado.edu