The triangular principle is equivalent to the triangular scheme

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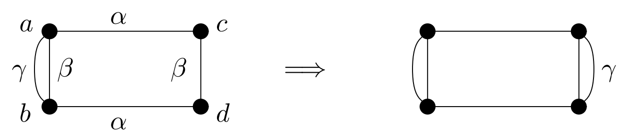
Abstract. The triangular scheme is a diagrammatic characterization of congruence join-semidistributivity. The triangular principle is a variant of this condition, where one of the congruences is replaced by a tolerance. This paper contains two proofs showing that the triangular principle and the triangular scheme are equivalent for varieties. The first one is a routine argument using tame congruence theory, and works only for locally finite varieties. The second proof is an elementary, but nontrivial calculation with terms, which works for arbitrary varieties. This yields a stronger Mal'tsev-condition characterizing congruence join-semidistributivity than the one obtained from the triangular scheme.

1. The history of the triangular scheme

A celebrated result of H. P. Gumm is the following characterization of congruence modularity. An algebra $\mathbf{A}$ satisfies the *shifting scheme* (sometimes called the shifting lemma), if for every $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ such that $\alpha \cap \beta \subseteq \gamma$, and any elements $a, b, c, d \in A$, the implication



holds. Writing $a\ \beta\ b$ to mean $(a, b) \in \beta$, this diagram asserts that, if $a\ \beta\ b$, $c\ \beta\ d$, $a\ \alpha\ c$, $b\ \alpha\ d$ and $a\ \gamma\ b$, then also $c\ \gamma\ d$. If we replace α by an arbitrary tolerance (reflexive, symmetric, compatible binary relation), then we get the *shifting principle*. Gumm proved in [8] that a variety is congruence modular if and only if every algebra in the variety satisfies the shifting scheme, or equivalently, if every algebra in the variety satisfies the shifting principle. This proved to be a useful tool for building up the theory of the modular commutator.

Is it possible to characterize congruence distributivity in a similar way? A natural candidate for such a diagram was proposed by I. Chajda and E. Horváth in [2].

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Definition 1.1. An algebra $\mathbf{A}$ satisfies the *triangular scheme* if for every choice of congruences $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ such that $\alpha \cap \beta \subseteq \gamma$, and every choice of elements $a, b, c \in A$, the implication

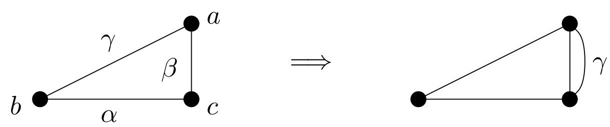


FIGURE 1. The triangular scheme

holds. That is, $b\ \alpha\ c$, $a\ \beta\ c$ and $a\ \gamma\ b$ imply that $a\ \gamma\ c$. The *triangular principle* is the same statement with the congruence α replaced by an arbitrary tolerance.

In a congruence distributive variety, the triangular scheme clearly holds. Indeed, if the congruence lattice of $\mathbf{A}$ is distributive, then, as noticed in [2], we have

$$(a, c) \in \beta \wedge (\alpha \vee \gamma) = (\beta \wedge \alpha) \vee (\beta \wedge \gamma) \subseteq \gamma\,.$$

In [2] it is also shown, using Jónsson-terms, that the triangular principle is satisfied in every congruence-distributive variety. In the papers [1, 2, 3, 4, 7] various sufficient conditions are provided, which ensure that a variety satisfying the triangular scheme is congruence distributive. Such sufficient conditions are congruence permutability, the shifting scheme (hence congruence modularity), or the so-called trapezoid lemma (which characterizes congruence distributivity by the results in [4]). However, the relationship between the triangular scheme, the triangular principle and congruence distributivity has remained open even for locally finite varieties.

The real meaning of the triangular scheme for locally finite varieties is revealed by a result of David Hobby and Ralph McKenzie. Theorem 9.11 of [9] states, among other things, the following.

Theorem 1.2. *For any locally finite variety $\mathcal{V}$, the following are equivalent (the numbering of the statements below follows that of* [9]*).*

(1) *$\mathcal{V}$ omits types* **1**, **2**, **5** *of tame congruence theory.*
(3) *If $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$, then*

$$\beta \cap (\alpha \circ \gamma) \subseteq \big(\alpha \vee (\beta \wedge \gamma)\big) \cap \big(\gamma \vee (\alpha \wedge \beta)\big)\,.$$

(4) *There exists an integer n and ternary terms $d_0, \ldots, d_{2n}$ such that $\mathcal{V}$ satisfies the following identities for $1 \le i \le n$:*

$$d_0(x, y, z) = x$$
$$d_{2i-2}(x, x, y) = d_{2i-1}(x, x, y)\,,$$
$$d_{2i-1}(x, y, x) = d_{2i}(x, y, x)\,, \quad d_{2i-1}(x, y, y) = d_{2i}(x, y, y)\,,$$
$$d_{2n}(x, y, z) = z\,.$$

(6) *The congruence lattice of every finite algebra in $\mathcal{V}$ is join-semidistributive.*

Condition (3) of this theorem clearly implies that the triangular scheme holds in $\mathbf{A}$. In fact it is easy to see the following statement for any (not necessarily locally finite) variety.

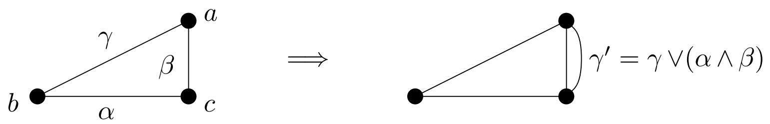
Proposition 1.3. *For any variety $\mathcal{V}$, the following are equivalent.*

(1) *Every algebra in $\mathcal{V}$ satisfies the triangular scheme.*
(2) *If $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$, then we have*

$$\beta \cap (\alpha \circ \gamma) \subseteq \gamma \vee (\alpha \wedge \beta)\,.$$

(3) *$\mathcal{V}$ satisfies the congruence-inclusion in condition (3) of Theorem 1.2.*
(4) *$\mathcal{V}$ satisfies the Mal'tsev-condition in (4) of Theorem 1.2.*

Proof. To prove the implication (1) $\implies$ (2) suppose that $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ are arbitrary, and $(c, a) \in \beta \cap (\alpha \circ \gamma)$. Then there exists an element $b \in \mathbf{A}$ such that $a\ \gamma\ b\ \alpha\ c$. Apply the triangular scheme for these elements with γ replaced by $\gamma' = \gamma \vee (\alpha \wedge \beta)$. We get that



holds, proving (2).

Now if (2) is true, then reversing the roles of α and γ we get that

$$\beta \cap (\gamma \circ \alpha) \subseteq \alpha \vee (\gamma \wedge \beta)\,.$$

By taking the converse of this inclusion and combining with (2) we obtain condition (3) of Theorem 1.2. Therefore the implication (2) $\implies$ (3) is proved.

To show that (3) $\implies$ (4) consider the free algebra in $\mathcal{V}$ generated by x, y, z, and define the congruences

$$\alpha = \mathrm{Cg}(y, z)\,, \quad \beta = \mathrm{Cg}(x, z)\,, \quad \gamma = \mathrm{Cg}(x, y)\,.$$

Then (the converse of) (3) implies that

$$(x, z) \in \gamma \vee (\alpha \wedge \beta)\,.$$

Using the standard technique for producing Mal'tsev-conditions it is trivial to see that we get the terms d_i of (4) above.

Finally to prove (4) $\implies$ (1) suppose that an algebra $\mathbf{A}$ has terms d_i satisfying (4). To prove that $\mathbf{A}$ satisfies the triangular scheme consider the elements $c_i = d_i(a, b, c)$. The identities in (4) imply that $c_j\ \gamma\ c_{j+1}$ when j is even, and $c_j\ (\alpha \wedge \beta)\ c_{j+1}$ when j is odd. Since $c_0 = a$, $c_{2n} = c$ and $\alpha \wedge \beta \subseteq \gamma$, we get that $(a, c) \in \gamma$ by transitivity. □

This observation shows immediately that the triangular scheme does not imply congruence distributivity, not even for locally finite varieties. Indeed, there are locally finite varieties satisfying the conditions of Theorem 1.2 that are not congruence distributive, for example, Polin's variety (see [14]). The fact that this variety has the properties we are claiming for it follows from Exercise 9.20 (6) of [9].

The congruences α, β and γ in the proof of (3) $\implies$ (4) satisfy $\alpha \vee \gamma = \beta \vee \gamma$. Hence if the congruence lattice of this free algebra is join-semidistributive, then we get that $(\alpha \wedge \beta) \vee \gamma = \alpha \vee \gamma$, which contains the pair (x, z), from which (4) follows. Therefore congruence join-semidistributivity implies the triangular scheme for any variety. The converse is also true, but it is difficult to prove. Generalizing (6) of Theorem 1.2 it is shown in [11] that the Mal'tsev-condition in (4) axiomatizes the class of congruence join-semidistributive varieties (for the locally finite case, this result has already appeared in [10], the first such Mal'tsev condition was given in [5]). This gives another explanation of the fact that congruence modular varieties satisfying the triangular scheme are congruence distributive (since a lattice is distributive if and only if it is modular and join-semidistributive). Thus the triangular scheme can (and should) be considered as a characterization of congruence join-semidistributivity.

The fact that the seemingly stronger shifting principle follows from the shifting scheme for varieties has proved useful in commutator theory. Similarly, G. Czédli and E. Horváth proved interesting results in [6] about congruence identities, using a modularity-like property of tolerances in a modular variety. Therefore it may be important to show that the triangular principle is equivalent to the triangular scheme for every variety. First we characterize the triangular principle.

Proposition 1.4. *For any variety $\mathcal{V}$, the following are equivalent.*

(1) *Every algebra in $\mathcal{V}$ satisfies the triangular principle.*
(2) *If $\mathbf{A} \in \mathcal{V}$, $\beta, \gamma \in \mathrm{Con}(\mathbf{A})$ and T is a tolerance of $\mathbf{A}$, then we have*

$$\beta \cap (T \circ \gamma) \subseteq \gamma \vee (T \cap \beta)^* ,$$

where $(T \cap \beta)^* \in \mathrm{Con}(\mathbf{A})$ denotes the transitive closure of $T \cap \beta$.
(4) *$\mathcal{V}$ satisfies the Mal'tsev-condition in (4) of Theorem 1.2, and in addition, there exist 5-ary terms e_i for $1 \le i \le n$ such that*

$$e_i(x, y, z, y, z) = d_{2i-1}(x, y, z) \quad \text{and} \quad e_i(x, y, z, z, y) = d_{2i}(x, y, z)$$

are also identities of $\mathcal{V}$ for every $1 \le i \le n$.

Proof. The proof is similar to the proof of Proposition 1.3, we only point out the differences. In (1) $\implies$ (2) the triangular principle is applied with $\gamma' = \gamma \vee (T \cap \beta)^*$. Notice that $\beta \cap (T \circ \gamma) \subseteq \gamma \vee (T \cap \beta)^*$ is equivalent to $\beta \cap (\gamma \circ T) \subseteq \gamma \vee (T \cap \beta)^*$ (by taking the converse of both sides). To show (2) $\implies$ (4) we let T be the tolerance of $\mathbf{F}_{\mathcal{V}}(x, y, z)$ generated by (y, z). This tolerance consists of the pairs

$$\big(e(x, y, z, y, z), e(x, y, z, z, y)\big) ,$$

where e is a 5-ary term. For (4) $\implies$ (1) note that $e(a, b, c, b, c)$ and $e(a, b, c, c, b)$ are related by any tolerance containing the pair (b, c). □

Of course the Mal'tsev-condition in (4) may be written using only the terms e_i, since both d_{2i-1} and d_{2i} can be expressed with e_i. However, such a formulation would obscure the relationship to (4) of Theorem 1.2.

There are several examples of results that have been proved using tame congruence theory in a locally finite setting, but have been later generalized to general varieties. These generalizations usually involve elaborate calculations with Mal'tsev-conditions. Examples of such results can be found in [11], [12] and [13]. In the next two sections we present another example of this phenomenon.

2. The locally finite case

In this section we give an easy proof of a strong version of our main theorem (Theorem 3.1), restricted to locally finite varieties. We do not know if the strong version holds for nonlocally finite varieties.

Theorem 2.1. *Let* $\mathbf{A}$ *be a finite algebra omitting types* $\mathbf{1}$, $\mathbf{2}$ *and* $\mathbf{5}$ *of tame congruence theory, and* $a, b, c \in A$. *Suppose that* S *and* T *are reflexive, compatible relations containing* (a, c) *and* (b, c), *respectively, and let* γ *be a congruence containing the pair* (a, b). *If* $S \cap T \subseteq \gamma$, *then* $(a, c) \in \gamma$.

Proof. The assertion of the theorem is the following version of the triangular principle. If $S \cap T \subseteq \gamma$, then

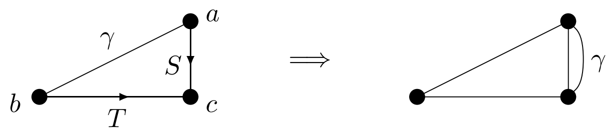


FIGURE 2. The strong triangular principle

In this figure, the fact that (a, c) is in S is indicated by a directed S-labeled edge from a to c, and similarly $(b, c) \in T$ is indicated by a directed T-labeled edge from b to c. (The directions are required because S and T need not be symmetric.) This version of the triangular principle is stronger than the usual version, because it allows S and T to be arbitrary reflexive, compatible relations rather than requiring S to be a congruence and T to be a tolerance. Although we are only proving that $(a, c) \in \gamma$, a symmetric argument shows also that $(b, c) \in \gamma$.

To prove the theorem, assume that $(a, c) \notin \gamma$. Let δ be a congruence of $\mathbf{A}$ that is maximal for $\gamma \subseteq \delta$ and $(a, c) \notin \delta$. Then δ is meet-irreducible, and its unique cover θ is $\delta \vee \mathrm{Cg}(a, c) = \delta \vee \beta$. Since $(a, c) \in \theta - \delta$, there exists a unary polynomial p of $\mathbf{A}$ that maps A to a $\langle \delta, \theta \rangle$-minimal set U such that $\big(p(a), p(c)\big) \notin \delta$. Then $\big(p(a), p(c)\big) \notin \gamma$ (as $\gamma \subseteq \delta$). Therefore we can replace a, b, c by $p(a)$, $p(b)$, $p(c)$, respectively, and this new configuration is still a failure of the implication in Figure 2, since p preserves S, T and γ.

Since $\mathbf{A}$ omits types $\mathbf{1}, \mathbf{2}$ and $\mathbf{5}$, the type of $\langle \delta, \theta \rangle$ is $\mathbf{3}$ or $\mathbf{4}$. Lemma 4.17 of [9] implies that in these types every minimal set U has a unique trace N, which has only two elements, and hence the restriction of δ to N is trivial. We know that $(a, c) \in \theta - \delta$, and therefore $N = \{a, c\}$. But $(a, b) \in \gamma \subseteq \delta \subseteq \theta$, and therefore the element b is also contained in $N = (a/\theta) \cap U$. Moreover, $(a, b) \in \delta$ implies that $a = b$. This is impossible, because then $(a, c) = (b, c)$ is contained in $S \cap T \subseteq \gamma$. This contradiction proves the theorem. $\square$

Corollary 2.2. *If every algebra of a locally finite variety $\mathcal{V}$ satisfies the triangular scheme, then every algebra of $\mathcal{V}$ satisfies the triangular principle, too.*

Proof. The triangular principle is formally stronger than the triangular scheme, so it suffices to derive the principle from the scheme. Let $\mathbf{B} \in \mathcal{V}$ be an algebra with elements a, b, and c related by congruences β and γ and a tolerance α, as indicated in the definition of the triangular scheme (see Figure 1). Replace $\mathbf{B}$ by its subalgebra $\mathbf{A}$ generated by $\{a, b, c\}$, replace γ by the restriction $\gamma|_{\mathbf{A}}$, and replace β and α by the compatible, reflexive relations S and T of $\mathbf{A}$ generated by (a, c) and (b, c), respectively. Then $\mathbf{A}, a, b, c, S, T$ satisfy the hypotheses of Theorem 2.1 with the congruence $\gamma|_{\mathbf{A}}$ playing the role of γ in that theorem. Indeed, the algebra $\mathbf{A}$ is finite, because it is finitely generated and $\mathcal{V}$ is locally finite, $\mathbf{A}$ omits types $\mathbf{1}, \mathbf{2}$ and $\mathbf{5}$, because $\mathcal{V}$ satisfies the triangular scheme (cf. Proposition 1.3 and Theorem 1.2), finally, $S \cap T \subseteq (\beta \cap \alpha) \cap A^2 \subseteq \gamma|_{\mathbf{A}}$. Thus Theorem 2.1 gives that $(a, c) \in \gamma|_{\mathbf{A}} \subseteq \gamma$, completing the proof. $\square$

3. The general case

In this section we prove the following.

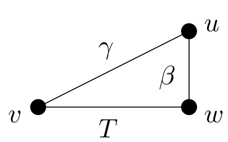
Theorem 3.1. *If every algebra of a variety $\mathcal{V}$ satisfies the triangular scheme, then every algebra of $\mathcal{V}$ satisfies the triangular principle, too.*

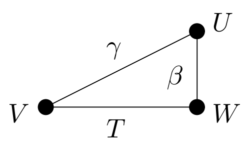
Suppose that the triangular scheme holds in every algebra of a variety $\mathcal{V}$. By Proposition 1.3, $\mathcal{V}$ satisfies the Mal'tsev-condition described in (4) of Theorem 1.2. We shall use these terms d_i in our calculations. For better readability, we shall sometimes omit the commas that normally separate arguments of terms.

We formulate the key lemma of the proof. Suppose that $\mathbf{A} \in \mathcal{V}$ has congruences α, β, a tolerance T, and elements u, v, w satisfying $u \;\gamma\; v \;T\; w \;\beta\; u$. Define $I = T \cap \beta$ and $\gamma' = \gamma \vee I^*$, where I^* denotes the transitive closure of I. Let f be a fixed idempotent ternary term of the algebra $\mathbf{A}$, and

$$U = f(w, v, u)\,, \quad V = f(w, v, v)\,, \quad W = f(w, v, w)\,.$$

These elements clearly satisfy the same relations, namely that $U \;\gamma\; V \;T\; W \;\beta\; U$. (We do not make the assumption that $I = T \cap \beta \subseteq \gamma$, because it is easier to follow the proof below this way.)





Lemma 3.2. *Suppose that g is also an idempotent ternary term of the algebra $\mathbf{A}$ satisfying the identities $f(x,y,x) = g(x,y,x)$ and $f(x,y,y) = g(x,y,y)$. Assume that $g(w,w,u)\ \gamma'\ u$ and $g(u,u,w)\ \gamma'\ w$. Then*

(1) $f(U,W,W)\ \gamma'\ w$;
(2) $f(W,W,U)\ \gamma'\ U$;
(3) $f(U,W,U)\ \gamma'\ u$;
(4) $f(U,U,W)\ \gamma'\ W$.

If in addition we have $U\ \gamma'\ W$, then $u\ \gamma'\ w$.

Proof. As g is idempotent, we have

$$g\big(g(wvw), g(wvw), g(wvw)\big) = g(wvw) = g\big(g(www), g(vvv), g(www)\big)\,. \tag{3.1}$$

Moving two instances of w to u we get β-related pairs:

$$g\big(g(wv\underline{u}), g(wvw), g(wvw)\big)\ \beta\ g\big(g(ww\underline{u}), g(vvv), g(www)\big)\,. \tag{3.2}$$

But $(v,w) \in T$, and therefore these two elements are related by $T \cap \beta = I \subseteq \gamma'$. By our assumption, $g(w,w,u)\ \gamma'\ u$ and $g(u,u,w)\ \gamma'\ w$. Hence the right side of (3.2) is

$$g\big(g(wwu), g(vvv), g(www)\big)\ \gamma'\ g(u,v,w)\ \gamma\ g(u,u,w)\ \gamma'\ w\,.$$

The identity $f(xyy) = g(xyy)$ implies that

$$g(wvu)\ \gamma\ g(wuu) = f(wuu)\ \gamma\ f(wvu) = U\,. \tag{3.3}$$

By the identity $f(xyx) = g(xyx)$ we have $g(wvw) = f(wvw) = W$, so the left side of equation (3.2) is

$$g\big(g(wvu), g(wvw), g(wvw)\big) = g\big(g(wvu), W, W\big)\ \gamma\ g(U,W,W)\,.$$

This is equal to $f(U,W,W)$ by the identity $f(xyy) = g(xyy)$. Hence (1) holds.

The identity $f(xyx) = g(xyx)$ implies that

$$\begin{aligned} &f\big(g(wvw), g(wvw), g(wvw)\big) = g(wvw) = f(wvw) = \\ &f\big(g(www), g(vvv), g(www)\big)\,. \end{aligned} \tag{3.4}$$

As in the previous paragraph, this implies that

$$f\big(g(wvw), g(wvw), g(wv\underline{u})\big)\ I\ f\big(g(www), g(vvv), g(ww\underline{u})\big)\,.$$

We assumed that $g(w,w,u)\ \gamma'\ u$, so the right side is γ'-related to $f(w,v,u) = U$. The left side is γ-related to $f(W,W,U)$, since $g(wvw) = f(wvw) = W$, and $g(wvu)\ \gamma\ U$ by (3.3). This proves (2).

From (3.4) we also get that

$$f\big(g(wv\underline{u}), g(wvw), g(wv\underline{u})\big)\ I\ f\big(g(ww\underline{u}), g(vvv), g(ww\underline{u})\big)\,.$$

The left side evaluates to $f(U,W,U)$ modulo γ (as $g(wvu)\ \gamma\ U$ and $g(wvw) = W$), and the right side to $f(u,v,u)$ modulo γ' (since $g(w,w,u)\ \gamma'\ u$), which is γ-related to $f(u,u,u) = u$. Hence (3) is true.

To prove (4) consider

$$\begin{aligned} &f\big(f(VWW), f(VWW), f(VWW)\big) = f(VWW) = \\ &f\big(f(VVV), f(WWW), f(WWW)\big)\,. \end{aligned}$$

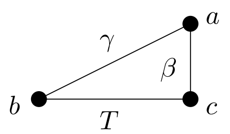
Therefore

$$f\big(f(VWW), f(VW\underline{U}), f(VWW)\big)\ I\ f\big(f(VVV), f(WW\underline{U}), f(WWW)\big)\,.$$

Now $f(VWW)\ \gamma\ f(UWW)\ \gamma'\ w$ by (1), and from (3) we get that $f(VWU)\ \gamma'\ u$. Finally, $f(WWU)\ \gamma'\ U$ by (2). Hence we obtain that $f(w,u,w)\ \gamma'\ f(V,U,W)$. But $f(wuw)\ \gamma\ f(wvw) = W$ and $f(VUW)\ \gamma\ f(UUW)$, so (4) holds.

To prove the last assertion of the lemma suppose that $U\ \gamma'\ W$. As f is idempotent, this implies, using (1), that $w\ \gamma'\ f(W,W,W) = W$. From (3) we get that $u\ \gamma'\ W$ with a similar argument. □

Now we can prove Theorem 3.1. Consider the usual situation



in an algebra $\mathbf{A} \in \mathcal{V}$. Define $I = T \cap \beta$ and $\gamma' = \gamma \vee I^*$ as above. We have to show that $(a,c) \in \gamma'$. Let $a_n = a$, $b_n = b$, $c_n = c$ and

$$a_{i-1} = d_{2i-1}(c_i, b_i, a_i)\,, \quad b_{i-1} = d_{2i-1}(c_i, b_i, b_i)\,, \quad c_{i-1} = d_{2i-1}(c_i, b_i, c_i)$$

for $n \geq i \geq 1$. Clearly, we have $a_i\ \gamma\ b_i\ T\ c_i\ \beta\ a_i$ for every i.

Claim 3.3. *We have $d_{2i}(c_i, c_i, a_i)\ \gamma'\ a_i$ and $d_{2i}(a_i, a_i, c_i)\ \gamma'\ c_i$ for $0 \leq i \leq n$.*

Proof. For $i = n$ we have equality. Suppose that the statement holds for i, we prove it for $i-1$. Apply Lemma 3.2 to $f = d_{2i-1}$, $g = d_{2i}$, $u = a_i$, $v = b_i$, $w = c_i$. Then $U = a_{i-1}$, $V = b_{i-1}$, $W = c_{i-1}$. The statement for i therefore says that $g(w,w,u)\ \gamma'\ u$ and $g(u,u,w)\ \gamma'\ w$. Thus the lemma implies that $f(W,W,U)\ \gamma'\ U$ and $f(U,U,W)\ \gamma'\ W$, that is,

$$d_{2i-1}(c_{i-1}, c_{i-1}, a_{i-1})\ \gamma'\ a_{i-1} \quad \text{and} \quad d_{2i-1}(a_{i-1}, a_{i-1}, c_{i-1})\ \gamma'\ c_{i-1}\,.$$

Hence the identity $d_{2i-2}(x,x,y) = d_{2i-1}(x,x,y)$ implies the statement for $i-1$. □

Claim 3.4. *We have $a_i\ \gamma'\ c_i$ for $0 \leq i \leq n$.*

Proof. Applying Claim 3.3 for $i = 0$ we get that that $c_0 = d_0(c_0, c_0, a_0)\ \gamma'\ a_0$. We induct on i, so suppose that $a_{i-1}\ \gamma'\ c_{i-1}$. The last statement of Lemma 3.2 (applied with the same parameters as in the previous proof) implies that $a_i\ \gamma'\ c_i$. □

Now take $i = n$ in the previous claim. It says that $a\ \gamma'\ c$, proving Theorem 3.1.

4. Concluding remarks

We wish to acknowledge the thorough work of the referee. He pointed out that the arguments in Section 3 are in fact arguments of commutator-theory, since in the proof of Lemma 3.2 we seem to use the term condition in the displayed formulas, where the underlined characters appear. Note also that in Gumm's original shifting lemma, one can actually replace the assumption $\alpha \wedge \beta \subseteq \gamma$ by $[\alpha, \beta] \subseteq \gamma$, and doing so results in a stronger theorem.

We did not endeavor to formulate a similar modification of the conclusion of Theorem 3.1 here. The main reason is that in any variety satisfying the triangular scheme, and hence the Mal'tsev condition of Theorem 1.2 (4), the usual TC-commutator of any two congruences is their intersection, thus no strengthening is obtained by proving a theorem from the hypothesis $[\alpha, \beta] \leq \gamma$ instead of $\alpha \wedge \beta \leq \gamma$. When working with tolerances rather than congruences, the commutator version of the result can actually be weaker than our version involving intersection. This depends on how the commutator of two tolerances is defined.

We show in [11] that a variety is congruence join-semidistributive if and only if it omits all abelian and so called rectangular congruences and tolerances. This allows us to use commutator theories to reduce questions to congruence join-semidistributive varieties, but no further. Congruence join-semidistributive varieties are those where the known commutator theories trivialize. We feel that the stronger Mal'tsev condition for congruence join-semidistributive varieties obtained in the present paper is likely to be of use only after such reductions are made.

We conclude the paper with a problem, related to the stronger version of the triangular property established in Theorem 2.1.

Problem 4.1. Is it true that if a variety $\mathcal{V}$ satisfies the triangular scheme, then it satisfies the stronger form of the triangular principle, where α is permitted to be an arbitrary reflexive, compatible (and not necessarily symmetric) relation?

Finally we mention that some of the easier arguments in this paper work for single algebras rather than varieties. We did not attempt to separate these arguments, or to formally state the corresponding results.

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