Strongly Abelian Varieties and the Hamiltonian Property *

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Abstract

In this paper we show that every locally finite strongly Abelian variety satisfies the Hamiltonian property. An algebra is Hamiltonian if every one of its subuniverses is a block of some congruence of the algebra. A counterexample is provided to show that not all strongly Abelian varieties are Hamiltonian.

1 Introduction

The class of strongly Abelian algebras was first defined by Ralph McKenzie in [6]. The significance of these algebras, especially in the role they play in the classification of finite algebras and locally finite varieties was demonstrated in [6, 4, 9]. Much of the work in this paper was motivated by the desire to understand the structure of locally finite Abelian varieties. In particular we would like to know whether or not every locally finite Abelian variety is Hamiltonian. This would answer a question posed in [4] and originally mentioned in [1].

In this paper we give an affirmative answer to this question for locally finite strongly Abelian varieties (Theorem 3.5). We also present an example that strongly Abelian varieties need not be Hamiltonian in general.

J. Shapiro in his doctoral thesis ([12] or see [13]) was able to settle the above problem under the assumption that the variety was not only locally

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finite and strongly Abelian, but also that there was only one fundamental operation in the language. He also obtained a strong structure theory for such varieties. A consequence of his structure theory is that every algebra in such a variety is quasi-affine. An algebra is said to be **quasi-affine** if it is a subalgebra of a reduct of an algebra that is polynomially equivalent to a module over some ring.

The situation is much more complicated when we allow more than one fundamental operation in the language, we provide examples which demonstrate this.

2 Definitions

The reader should consult [2] for general background information on universal algebra.

Definition 2.1

(1) An algebra is called **Abelian** if for all terms $t(x, \bar{y})$, for all a, b, $\bar{c}$ and $\bar{d}$,

$$t(a, \bar{c}) = t(a, \bar{d}) \longrightarrow t(b, \bar{c}) = t(b, \bar{d}).$$

(2) An algebra is called **strongly Abelian** if for all terms $t(x, \bar{y})$, for all a, b, $\bar{c}$, $\bar{d}$ and $\bar{e}$,

$$t(a, \bar{c}) = t(b, \bar{d}) \longrightarrow t(a, \bar{e}) = t(b, \bar{e}).$$

(3) An algebra is called **Hamiltonian** if every nonempty subuniverse is a block of some congruence of the algebra.

(4) For $f(x_1, \ldots, x_n)$ a function on a set A, we say that f depends on the variable x_i if there are elements $a_1, \ldots, a_n$ and b from A such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

Let $\epsilon(f)$ equal the essential arity of f, i.e., the number of variables on which f depends. For $\mathbf{A}$ an algebra, let

$$\epsilon(\mathbf{A}) = \max\{\epsilon(t^{\mathbf{A}}) \, : \, t \text{ is a term of } \mathbf{A}\}$$

and for $\mathcal{V}$ a variety, let

$$\epsilon(\mathcal{V}) = \epsilon(\mathbf{F}_{\mathcal{V}}(\omega)).$$

For $t(x_1, \ldots, x_n)$ a term of $\mathcal{V}$, let

$$\epsilon(t) = \epsilon(t^{\mathbf{F}_{\mathcal{V}}(\omega)}).$$

If $\mathcal{V}$ is a variety and every algebra in $\mathcal{V}$ is (P), where (P) is one of the properties defined above, then we say that $\mathcal{V}$ is (P).

The Abelian property is a generalization of what it means for a group to be Abelian, i.e., a group is Abelian in the above sense if and only if its multiplication is commutative. It is easy to see that any module over a ring is Abelian too. Another kind of Abelian structure is a multi-unary algebra. This is an algebra where all of the fundamental operations are unary. In fact such structures are strongly Abelian. An example of a non-unary strongly Abelian algebra is a rectangular band, i.e., a semigroup which satisfies the identity $xyx \approx x$. We will provide further examples in a later section.

It is not hard to see that a group is Hamiltonian if and only if every one of its subgroups is normal. While there are examples of non-Abelian, Hamiltonian groups, it turns out that a variety of groups is Hamiltonian if and only if it is Abelian. We shall see in this paper that for certain kinds of varieties this equivalence holds but tha t in general not all Abelian varieties are Hamiltonian.

We now state several well known facts about Hamiltonian and strongly Abelian algebras.

PROPOSITION 2.2 *Let* $\mathbf{A}$ *be an algebra and* $\mathcal{V}$ *a variety. Then the following hold:*

(i) *If* $\mathbf{A}^2$ *is Hamiltonian, then* $\mathbf{A}$ *is Abelian.*

(ii) *Let* $\mathcal{V}$ *be Hamiltonian. Then* $\mathcal{V}$ *is Abelian.*

PROOF. It is not hard to show that an algebra $\mathbf{A}$ is Abelian if and only if the diagonal subuniverse 0_A of $\mathbf{A}^2$ is a block of some congruence on $\mathbf{A}^2$. Part (i) follows from this fact, and (ii) is immediate from (i). ∎

THEOREM 2.3 *Let* $\mathbf{A}$ *be a finite algebra.*

(i) *The following are equivalent:*

a) $\mathbf{A}$ *is strongly Abelian.*

b) *For all terms* $t(x_1, \ldots, x_n)$ *there exists equivalence relations* R_1 ,..., R_n *on* A *such that for all* $\bar{a}$ *and* $\bar{b}$ *from* A,

$$t^{\mathbf{A}}(\bar{a}) = t^{\mathbf{A}}(\bar{b})$$

if and only if

$$\langle a_i, b_i \rangle \in R_i$$

for all $i \leq n$.

c) *For all terms* $t(x, \bar{y})$, *for all* a, b, e, $\bar{c}$, *and* $\bar{d}$,

$$t(a, \bar{c}) = t(b, \bar{d}) \longrightarrow t(e, \bar{c}) = t(e, \bar{d}).$$

(ii) *Let* $\mathbf{A}$ *be a strongly Abelian algebra. Then* $\epsilon(\mathbf{A})$ *is finite.*

(iii) *Let* $\mathcal{V}$ *be a locally finite strongly Abelian variety. Then* $\mathcal{V}$ *is finitely generated. Furthermore,* $\epsilon(\mathcal{V})$ *is finite.*

PROOF. The proof of (i) is elementary and the equivalence between a) and b) was first noted in [6, Lemma 2.6]. The proofs of (ii) and (iii) follow from (i) (see also [9, Theorem 0.17]). ∎

The following characterization of Hamiltonian varieties is due to Klukovits ([5]).

THEOREM 2.4 *A variety* $\mathcal{V}$ *is Hamiltonian if and only if for all terms* $t(x_1, \ldots, x_n)$ *of* $\mathcal{V}$, *there is a term* $r_t(x, y, z)$ *such that*

$$\mathcal{V} \models r_t(t(x_1, \ldots, x_n), x_1, z) \approx t(z, x_2, \ldots, x_n).$$

It is easy to show that if in the above theorem $\mathcal{V}$ is also strongly Abelian then the term r_t does not depend on its second variable. Corollary 3.6 gives us a characterization of locally finite strongly Abelian varieties that is similar to the above characterization of Hamiltonian varieties.

3 Locally Finite Strongly Abelian Varieties

In this section we will prove that every locally finite strongly Abelian variety is Hamiltonian. In fact we will show that if $\mathcal{V}$ is a strongly Abelian variety with $\epsilon(\mathcal{V})$ finite, then $\mathcal{V}$ is Hamiltonian. Throughout this section, let $\mathcal{V}$ be such a variety.

LEMMA 3.1 *For any term $t(x, \bar{y})$ of $\mathcal{V}$, there is a term $s(z, x, \bar{y})$ such that s depends on its first variable and*

$$\mathcal{V} \models s(t(x, \bar{y}), x, \bar{y}') \approx t(x, \bar{y}').$$

PROOF. Let $\theta = \mathrm{Cg}_{\mathbf{F}}(t(x, \bar{y}), t(w, \bar{y}))$ in $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, w, \bar{y}, \bar{y}')$. Since $\mathcal{V}$ is strongly Abelian, then

$$\langle t(x, \bar{y}'), t(w, \bar{y}') \rangle \in \theta.$$

It then follows, using Mal'cev's characterization of principal congruences that there is a term $r(z, x, w, \bar{y}, \bar{y}')$ such that r depends on its first variable and either

$$r(t(x, \bar{y}), x, w, \bar{y}, \bar{y}') = t(x, \bar{y}')$$

or

$$r(t(w, \bar{y}), x, w, \bar{y}, \bar{y}') = t(x, \bar{y}').$$

In either case, it follows, since $\mathbf{F}$ is strongly Abelian, that the term r does not depend on the variables w or $\bar{y}$. Let $s(z, x, \bar{y})$ be the term obtained from r by disregarding these variables. So the first of the above equations implies that

$$\mathcal{V} \models s(t(x, \bar{y}), x, \bar{y}') \approx t(x, \bar{y}')$$

as required and from the second equation we have

$$\mathcal{V} \models s(t(w, \bar{y}), x, \bar{y}') \approx t(x, \bar{y}').$$

By identifying the variable w with x, we get the desired equation. ∎

LEMMA 3.2 *For every term $t(x, \bar{y})$ of $\mathcal{V}$, there is a term $r_t(x, \bar{y})$ such that $\epsilon(t) = \epsilon(r_t)$ and*

$$\mathcal{V} \models r_t(t(x, \bar{y}), \bar{y}') \approx t(x, \bar{y}').$$

PROOF. We argue by contradiction. Choose a term $t(x, \bar{y})$ with $\epsilon(t)$ maximal such that no term r_t as in the statement of the lemma exists. Without loss of generality, assume that t depends on all of its variables. By lemma 3.1, there is a term $s(z, x, \bar{y})$ such that s depends on z and

$$\mathcal{V} \models s(t(x, \bar{y}), x, \bar{y}') \approx t(x, \bar{y}').$$

If s does not depend on the variable x, we can set $r_t(x, \bar{y}) = s(x, x, \bar{y})$.

If s does depend on x, then $\epsilon(s) = \epsilon(t) + 1$ since s must depend on all of the variables in $\bar{y}$. By our choice of t it follows that there is a term r_s which depends on all of its variables such that

$$\mathcal{V} \models r_s(s(z, x, \bar{y}), z', \bar{y}') \approx s(z', x, \bar{y}').$$

Setting $r_t(x, \bar{y}) = r_s(x, x, \bar{y})$, it is not hard to show that

$$\mathcal{V} \models r_t(t(x, \bar{y}), \bar{y}') \approx t(x, \bar{y}')$$

as required. ∎

Definition 3.3

(1) An operation $d(x_1, \ldots, x_n)$ on a set A is called a **diagonal operation** for the operation $t(x_1, \ldots, x_n)$ if for all x_i^j from A,

$$d(t(x_1^1, \ldots, x_n^1), \ldots, t(x_1^n, \ldots, x_n^n)) = t(x_1^1, x_2^2, \ldots, x_n^n).$$

An operation d is called **diagonal** if it is a diagonal operation for itself.

A term $d(\bar{x})$ is called a **diagonal term** for the term $t(\bar{x})$ in the variety $\mathcal{V}$ if for all $\mathbf{A} \in \mathcal{V}$, the operation $d^{\mathbf{A}}$ is diagonal for $t^{\mathbf{A}}$. A term d is called **diagonal** in $\mathcal{V}$ if it is a diagonal term for itself.

(2) An operation d on a set A is called **diagonal for a subset** B if

$$d(d(x_1^1, \ldots, x_n^1), \ldots, d(x_1^n, \ldots, x_n^n)) = d(x_1^1, \ldots, x_n^n)$$

and

$$d(x, \ldots, x) = x$$

for all x, x_i^j in B.

A term d is called a **diagonal term for the subset** B of the algebra $\mathbf{A}$ if the operation $d^{\mathbf{A}}$ is diagonal for B.

(3) An operation $d(x_1, \ldots, x_n)$ on a set $B = B_1 \times \cdots \times B_n$ is called a **decomposition operation for** B if

$$d(\vec{b}_1, \ldots, \vec{b}_n) = \langle b_1^1, \ldots, b_n^n \rangle$$

for all $\vec{b}_i = \langle b_1^i, \ldots, b_n^i \rangle$ in B.

LEMMA 3.4 *Let $t(x_1, \ldots, x_n)$ be a term of $\mathcal{V}$. There is a diagonal term for t.*

PROOF. We define a sequence of terms q_i, $1 \leq i \leq n+1$ as follows: let $q_1 = t$ and given q_i, let q_{i+1} be such that

$$\begin{aligned} \mathcal{V} \quad \models \quad & q_{i+1}(x_1, \ldots, x_{i-1}, q_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n), \\ & x_{i+1}, \ldots, x_n) \approx q_i(x_1, \ldots, x_n). \end{aligned}$$

The existence of such terms is guaranteed by the previous lemma.

We now prove by induction that for each $i \leq n+1$ we have

$$q_i(t(x_1^1, \ldots, x_n^1), \ldots, t(x_1^{i-1}, \ldots, x_n^{i-1}), y_i, y_{i+1}, \ldots, y_n)$$

$$\approx t(x_1^1, x_2^2, \ldots, x_{i-1}^{i-1}, y_i, y_{i+1}, \ldots, y_n).$$

This is clear for $i = 1$. Given that the above is true for i, we have,

$$\begin{aligned} & q_{i+1}(t(x_1^1, \ldots, x_n^1), \ldots, t(x_1^i, \ldots, x_n^i), y_{i+1}, y_{i+2}, \ldots, y_n) \\ \approx \quad & q_{i+1}(t(x_1^1, \ldots, x_n^1), \ldots, t(x_1^{i-1}, \ldots, x_n^{i-1}), \\ & q_i(t(x_1^i, \ldots, x_n^i), \ldots, t(x_1^i, \ldots, x_n^i), x_i^i, \ldots, x_n^i), y_{i+1}, \ldots, y_n) \\ \approx \quad & q_i(t(x_1^1, \ldots, x_n^1), \ldots, t(x_1^{i-1}, \ldots, x_n^{i-1}), x_i^i, y_{i+1}, \ldots, y_n) \\ \approx \quad & t(x_1^1, x_2^2, \ldots, x_{i-1}^{i-1}, x_i^i, y_{i+1}, \ldots, y_n). \end{aligned}$$

Finally it should be clear that the term q_{n+1} is a diagonal term for t. ∎

THEOREM 3.5 *Let $\mathcal{V}$ be a strongly Abelian variety with $\epsilon(\mathcal{V})$ finite. Then $\mathcal{V}$ is Hamiltonian.*

PROOF. We will use the condition from Theorem 2.4 to prove this theorem. For $t(x, \bar{y})$ a term of $\mathcal{V}$, let

$$r_t(x, y) = d_t(t(y, y, \ldots, y), x, x, \ldots, x),$$

where d_t is some diagonal term for t. Then

$$\mathcal{V} \models r_t(t(x, \bar{y}), z) \approx t(z, \bar{y})$$

as required. ∎

COROLLARY 3.6 *Let $\mathcal{V}$ be a locally finite variety (or with $\epsilon(\mathcal{V})$ finite). Then $\mathcal{V}$ is strongly Abelian if and only if for all terms $t(x, \bar{y})$ there is a term $r_t(x, y)$ such that*

$$\mathcal{V} \models r_t(t(x, \bar{y}), z) \approx t(z, \bar{y}).$$

PROOF. One direction of this Corollary follows immediately from the proof of Theorem 3.5. For the other, let $t(x, \bar{y})$ be a term of $\mathcal{V}$ and let $\mathbf{A}$ be a member of $\mathcal{V}$. Suppose that we have

$$t^{\mathbf{A}}(a, \bar{c}) = t^{\mathbf{A}}(b, \bar{d})$$

and $e \in A$.

Then

$$t^{\mathbf{A}}(e, \bar{c}) = r_t^{\mathbf{A}}(t^{\mathbf{A}}(a, \bar{c}), e) = r_t^{\mathbf{A}}(t^{\mathbf{A}}(b, \bar{d}), e) = t^{\mathbf{A}}(e, \bar{d}).$$

By Theorem 2.3, this is sufficient to show that $\mathbf{A}$ is strongly Abelian. ∎

COROLLARY 3.7 *Let $\mathcal{V}$ be a strongly Abelian variety such that $\mathbf{F}_{\mathcal{V}}(2)$ is finite. Then $\mathcal{V}$ is Hamiltonian. Thus if $\mathcal{V}$ is strongly Abelian and locally finite, it is Hamiltonian.*

PROOF. The proof of Theorem 0.17 of [9] shows that under the above assumptions $\epsilon(\mathcal{V})$ is finite. ∎

4 Diagonal Operations

Diagonal terms were studied by Płonka in [10] and have been used in the investigation of direct decompositions (see [8]). More recently, diagonal terms have been used by McKenzie in [6] to analyze the structure of finite minimal strongly Abelian algebras having one fundamental operation. Valeriote also used them in [16] to uncover the structure of locally finite decidable strongly Abelian varieties. We now present some elementary properties of diagonal operations and then use them to analyze algebras in locally finite strongly Abelian varieti es.

Definition 4.1 Let $\mathbf{A}$ be an algebra. The **clone of A**, denoted by Clo $\mathbf{A}$ is the collection of all term operations on $\mathbf{A}$. $\mathrm{Clo_n}\mathbf{A}$ will denote the collection of all n-ary members of Clo $\mathbf{A}$. The clone of $\mathbf{A}$ can be characterized as the smallest collection of operations on A that contains the projection operations and the fundamental operations of $\mathbf{A}$ and is closed under composition.

Recalling Definition 3.3 we will write $d \geq f$ if d is a diagonal operation for the operation f.

PROPOSITION 4.2 *Let* $\mathbf{A}$ *be an algebra and let* d *and* f *belong to* $\mathrm{Clo_n}\mathbf{A}$.

(i) *If* $d \geq f$ *then* d *is a diagonal operation for the range of* f.

(ii) *Let* d *be diagonal. Then* $d \geq f$ *if and only if*

$$f(x_1, \ldots, x_n) = d(f_\delta(x_1), \ldots, f_\delta(x_n))$$

for all x_i, *where* $f_\delta(x)$ *is the operation* $f(x, \ldots, x)$.

(iii) *The relation* $\geq$ *is transitive and antisymmetric on* Clo $\mathbf{A}$.

(iv) *If* d *is a diagonal operation, i.e.,* $d \geq d$, *then* $d \geq d(g(x_1), \ldots, g_n(x_n))$ *for all unary functions* g_i *on* A.

PROOF.

(i) Let B be the range of f. If $a \in B$, then $a = f(a_1, \ldots, a_n)$ for some a_i in A. Thus

$$\begin{aligned} d(a, \ldots, a) &= d(f(a_1, \ldots, a_n), \ldots, f(a_1, \ldots, a_n)) \\ &= f(a_1, \ldots, a_n) = a. \end{aligned}$$

If $b_i^j \in B$ for $i, j \leq n$ we must show that

$$d(d(b_1^1, \ldots, b_n^1), \ldots, d(b_1^n, \ldots, b_n^n)) = d(b_1^1, \ldots, b_n^n).$$

This easily follows since each of the b_i^j are in the range of f and d is diagonal for f.

(ii) Suppose that $d \geq f$. Then

$$f(x_1, \ldots, x_n) = d(f(x_1, \ldots, x_1), \ldots, f(x_n, \ldots, x_n))$$

as required.

Conversely, suppose that

$$f(x_1, \ldots, x_n) = d(f_\delta(x_1), \ldots, f_\delta(x_n)).$$

Then since d is diagonal

$$\begin{aligned} & d(f(x_1^1, \ldots, x_n^1), \ldots, f(x_1^n, \ldots, x_n^n)) \\ &= d(d(f_\delta(x_1^1), \ldots, f_\delta(x_n^1)), \ldots, d(f_\delta(x_1^n), \ldots, f_\delta(x_n^n))) \\ &= d(f_\delta(x_1^1), \ldots, f_\delta(x_n^n)) \\ &= f(x_1^1, \ldots, x_n^n) \end{aligned}$$

establishing that $d \geq f$.

(iii) The transitivity of $\geq$ is not difficult to prove. For antisymmetry, suppose that $f \geq d$ and $d \geq f$. Then $f(x_1, \ldots, x_n) = d(f_\delta(x_1), \ldots, f_\delta(x_n))$ and $f_\delta(x_i) = x_i$ yields $f = d$.

(iv) We leave this as an exercise.

∎

COROLLARY 4.3 *Let $\mathcal{V}$ be a strongly Abelian variety with $\epsilon(\mathcal{V})$ finite and let $\mathbf{A}$ belong to $\mathcal{V}$. Then* Clo $\mathbf{A}$ *is generated by a set of diagonal opera tions along with a set of unary functions. If $\mathcal{V}$ is locally finite, then we can choose these generating sets to be finite.*

PROOF. The first part of this corollary follows from Lemma 3.4 and the previou s proposition. If $\mathcal{V}$ is locally finite, then being strongly Abelian it is in fact finitely generated, say by the finite algebra $\mathbf{F}$. It will suffice to show that Clo $\mathbf{F}$ is generated by a finite number of diagona l operations and unary functions. This follows easily from the first part of this corollary and from $\epsilon(\mathbf{F})$ being finite. ∎

It is conceivable that one could completely characterize those finite algebras that generate strongly Abelian varieties by describing how the finitely many diagonal and unary operations provided by the previous corollary must "fit together". Certainly these operations must be highly compatible. We do not pursue this possibility in this paper.

A significant feature of diagonal terms is stated in the following lemma whose proof can be found in [9, Lemma 11.4].

LEMMA 4.4 *Let $d(x_1, \ldots, x_n)$ be a diagonal term for the subset B of the strongly Abelian algebra $\mathbf{A}$. Then there is an algebra $\mathbf{A}'$ isomorphic to $\mathbf{A}$ such that the image of B, call it B', under this isomorphism is equal to the cartesian product*

$$B_1 \times \cdots \times B_n$$

for some sets $B_1, \ldots, B_n$ and such that

$$d^{\mathbf{A}'}(\bar{b}_1, \ldots, \bar{b}_n) = \langle b_1^1, \ldots, b_n^n \rangle$$

for all $\vec{b}^i = \langle b_1^i, \ldots, b_n^i \rangle$ in B', i.e., $d^{\mathbf{A}'}|_{B'}$ is a decomposition operation for $B_1 \times \cdots \times B_n$. Furthermore, the set B_i will have more than one element if and only if the function $d^{\mathbf{A}}|_B$ depends on its ith variable.

We now show that if $\mathcal{V}$ is strongly Abelian with $\epsilon(\mathcal{V})$ finite, then locally the members of Clo $\mathbf{A}$ can be regarded as sequences of unary operations, for any algebra $\mathbf{A}$ in $\mathcal{V}$.

Definition 4.5

(1) Let $f(x_1, \ldots, x_m)$ be an operation from a set $B_1 \times \cdots \times B_k$ to a set $C_1 \times \cdots \times C_n$. We say that f **factors** (into unary functions) if there is a (not necessarily unique) sequence of unary functions

$$f_1 : B_{i_1} \longrightarrow C_1, \ldots, f_n : B_{i_n} \longrightarrow C_n$$

such that

$$f(\vec{a}^1, \ldots, \vec{a}^m) = \langle f_1(a_{i_1}^{j_1}), \ldots, f_n(a_{i_n}^{j_n}) \rangle$$

for some $i_l \leq k$ and $j_p \leq m$, where for $q \leq m$, $\vec{a}^q = \langle a_1^q, a_2^q, \ldots, a_k^q \rangle$.

(2) Let $d_1(x_1, \ldots, x_k)$ and $d_2(x_1, \ldots, x_n)$ be diagonal operations for the sets B and C respectively and let let $f(x_1, \ldots, x_m)$ be a map from B^m to C. We say that f **factors with respect to** d_1 and d_2 if f factors when considered as a map from $B_1 \times \cdots \times B_k$ to $C_1 \times \cdots \times C_n$, where these cartesian products are those induced (as in Lemma 4.4) by the operations d_1 and d_2 on B and C respectively.

THEOREM 4.6 *Let $\mathcal{V}$ be a strongly Abelian variety with $\epsilon(\mathcal{V})$ finite and let $\mathbf{A}$ belong to $\mathcal{V}$. If $f(x_1, \ldots, x_m) \in \mathrm{Clo}\,\mathbf{A}$ and d is a diagonal term operation for the subset B of A, then there is some term operation d' of $\mathbf{A}$ that is diagonal for the set $f(B^m)$ and such that $f|_B : B \longrightarrow f(B^m)$ factors with respect to $d|_B$ and $d'|_{f(B^m)}$.*

PROOF. We may assume that f and d depend on all of their variables. Choose d' in $\mathrm{Clo}\,\mathbf{A}$ depending on all of its variables such that d' is diagonal on the set $f(B^m)$ and of maximal arity.

Consider the term

$$\begin{aligned} d'(f(d(x_{1,1}^1, \ldots, x_{1,k}^1), \ldots, d(x_{m,1}^1, \ldots, x_{m,k}^1)), \ldots, \\ f(d(x_{1,1}^n, \ldots, x_{1,k}^n), \ldots, d(x_{m,1}^n, \ldots, x_{m,k}^n))) \end{aligned} \tag{1}$$

where the arity of d is k and the arity of d' is n. Since d' is diagonal for $f(B^m)$ then it is not hard to see that the range of term (1) contains $f(B^m)$. Thus by our choice of d' it follows that term (1) can depend on at most n variables. (Otherwise by Lemma 3.4 we would have some diagonal term of higher essential arity for the set $f(B^m)$.)

In fact, it can be shown that for each $i \leq n$ there is at most one $\{j, r\}$ such that the above term depends on the variable $x_{j,r}^i$. If, for example, the above term depends on $x^1_{j,r}$ and $x^1_{u,v}$ where $\{j,r\}$ and $\{u,v\}$ are different, then the following term would depend on at least $n+1$ variables and have range containing $f(B^m)$, contrary to our choice of d'.

$$d'(f(d(x^1_{1,1},\dots,x^1_{1,k}),\dots,d(x^1_{m,1},\dots x^1_{m,k})),y_2,\dots,y_n)$$

So there is a function $\sigma : \{1,\dots,n\} \longrightarrow \{1,\dots,m\} \times \{1,\dots,k\}$ such that the term (1) depends on at most the variables $\{x^1_{\sigma(1)},\dots,x^n_{\sigma(n)}\}$.

We have seen in Lemma 4.4 that the terms d and d' can be used to decompose the sets B and $f(B^m)$ into factors $B_1 \times \dots \times B_k$ and $B'_1 \times \dots \times B'_n$ so that d and d' act as decomposition operations when restricted to these sets. If we regard $f|_B$ now as a map from $(B_1 \times \dots \times B_k)^m$ into $B'_1 \times \dots \times B'_n$, then we have

$$f|_B(\vec{b}_1,\dots,\vec{b}_m) = \langle f_1(b_{1,1},\dots,b_{m,k}),\dots,f_n(b_{1,1},\dots,b_{m,k})\rangle$$

where $f_i : (B_1 \times \dots \times B_k)^m \longrightarrow B'_i$ is the projection of $f|_B$ onto B'_i, and

$$\vec{b}_i = \langle b_{i,1},\dots,b_{i,k}\rangle$$

for $i \leq n$.

From our observations in the previous paragraphs, it follows that the function $f_i(x_{1,1},\dots,x_{m,k})$ can depend on at most the variable $x_{\sigma(i)}$. But then we have shown that $f|_B$ factors with respect to $d|_B$ and $d|_{f(B^m)}$ as required. ∎

5 A non-Hamiltonian example

In this section we present an algebra $\mathbf{A}$ such that the variety generated by $\mathbf{A}$ is strongly Abelian, but not Hamiltonian. We shall define an appropriate commutative ring $\mathbf{R}$, and $\mathbf{A}$ will be a reduct of the $\mathbf{R}$-module $\mathbf{M} = {}_{\mathbf{R}}\mathbf{R}$. Thus our first aim is to find strongly Abelian reducts of modules.

Definition 5.1

(1) Let $\mathbf{R}$ be a commutative ring with identity. A family $\{a_1,\dots,a_n\} \subseteq R$ is called **orthogonal** with respect to $\tilde{a}_1,\dots,\tilde{a}_n \in R$ if $\tilde{a}_i a_j = \delta_{ij} a_j$ holds for all $1 \leq i,j \leq n$, where δ_{ij} is the Kronecker-delta symbol. The family $\{a_1,\dots,a_n\}$ of R is called **orthogonal** if it is orthogonal with respect to suitable elements of R. Note that if $\{a_1,\dots,a_n\}$ is orthogonal, then $a_i a_j = 0$ for $i \neq j$.

(2) Let $\mathbf{M}$ be a left $\mathbf{R}$-module. Define $\mathrm{Ort}(\mathbf{M})$ to be the set

$$\left\{\sum_{i=1}^{n} \lambda_i x_i \in \mathrm{Clo}\,\mathbf{M} \mid \{\lambda_1, \ldots, \lambda_n\} \subseteq R \text{ orthogonal}, \ n \in \omega\right\}.$$

LEMMA 5.2 *Let* $\mathbf{R}$ *be a commutative ring with identity and let* $\mathbf{M}$ *be a left* $\mathbf{R}$*-module. Then* $\mathrm{Ort}(\mathbf{M})$ *is a strongly Abelian clone on* M*.*

PROOF. Let $C = \mathrm{Ort}(\mathbf{M})$. The projections are contained in C, since the family $\{0, \ldots, 0, 1, 0, \ldots, 0\}$ is orthogonal (with respect to itself). To prove that C is closed under composition, let $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i x_i$ and for $1 \leq i \leq n$ let $g_i(x_1, \ldots, x_k) = \sum_{j=1}^{k} \mu_{ij} x_j$. Then for $h = f(g_1, \ldots, g_n)$ we have $h(x_1, \ldots, x_k) = \sum_{j=1}^{k} \nu_j x_j$, where

$$\nu_j = \sum_{i=1}^{n} \lambda_i \mu_{ij} \qquad (1 \leq j \leq k).$$

If $\{\lambda_1, \ldots, \lambda_n\}$ is orthogonal with respect to $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n\}$ and $\{\mu_{i1}, \ldots, \mu_{ik}\}$ is orthogonal with respect to $\{\tilde{\mu}_{i1}, \ldots, \tilde{\mu}_{ik}\}$, then an easy calculation shows that $\{\nu_1, \ldots, \nu_k\}$ is orthogonal with respect to $\{\tilde{\nu}_1, \ldots, \tilde{\nu}_k\}$, where

$$\tilde{\nu}_j = \sum_{i=1}^{n} \tilde{\lambda}_i \tilde{\mu}_{ij} \qquad (1 \leq j \leq k).$$

Thus C is a clone indeed.

To prove that C is strongly Abelian, let $f \in C$, $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i x_i$, where $\{\lambda_1, \ldots, \lambda_n\}$ is orthogonal with respect to $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n\}$. Suppose that $f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n)$. Multiplying by $\tilde{\lambda}_i$ we obtain that $\lambda_i a_i = \lambda_i b_i$ for $1 \leq i \leq n$. Hence for every x we have $f(x, a_2, \ldots, a_n) = f(x, b_2, \ldots, b_n)$, so C is indeed strongly Abelian. ∎

For the actual construction of $\mathbf{A}$ we need a clone that is smaller than the full orthogonal clone on $\mathbf{M}$.

LEMMA 5.3 *Let* $\mathbf{R}$ *be a commutative ring with identity,* $\mathbf{M}$ *a left* $\mathbf{R}$*-module, and* $\mathcal{I}$ *a set of ideals of* $\mathbf{R}$*. Then*

$$C_{\mathcal{I}}(\mathbf{M}) = \left\{\sum_{i=1}^{n} \lambda_i x_i \in \mathrm{Clo}\,\mathbf{M} \mid (\exists I \in \mathcal{I})(\lambda_1, \ldots, \lambda_n \in I)\right\} \cup \{\text{all projections}\}$$

is a clone on M*.*

PROOF. Assume $f, g_1, \ldots, g_n \in C = C_{\mathcal{I}}(\mathbf{M})$, where f is n-ary, and let $h = f(g_1, \ldots, g_n)$. We have to prove that $h \in C$. If f is a projection, then this is obvious. If not, then $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_1, \ldots, \lambda_n \in I \in \mathcal{I}$. Now it is straightforward to see (by looking at the formulas in the previous proof) that the coefficients of h are also contained in I. ∎

We now have a technique of finding strongly Abelian clones. We need, however, algebras that generate strongly Abelian varieties.

LEMMA 5.4 *Let* $\mathbf{A}$ *be an algebra such that for every* $f \in \mathrm{Clo}_{1+k}\mathbf{A}$ *there exist* $p_0, \ldots, p_n \in \mathrm{Clo}_{4+2k}\mathbf{A}$ *satisfying the following identities:*

$$\begin{aligned} p_0(x, y, z, \bar{u}, \bar{v}, f(x, \bar{u})) &= f(z, \bar{u}) \\ p_i(x, y, z, \bar{u}, \bar{v}, f(y, \bar{v})) &= p_{i+1}(x, y, z, \bar{u}, \bar{v}, f(x, \bar{u})) \qquad (0 \leq i < n) \\ p_n(x, y, z, \bar{u}, \bar{v}, f(y, \bar{v})) &= f(z, \bar{v})\,. \end{aligned}$$

Then $\mathbf{A}$ *generates a strongly Abelian variety.*

PROOF. These identities hold in the variety generated by $\mathbf{A}$ as well, and if $f(a, \bar{c}) = f(b, \bar{d})$ holds in any algebra satisfying these identities, then by substituting a, b, e, $\bar{c}$, $\bar{d}$ for x, y, z, $\bar{u}$, $\bar{v}$ they clearly imply $f(e, \bar{c}) = f(e, \bar{d})$. ∎

Next we investigate the meaning of the above identities in clones of modules. In order to avoid messy formulas we introduce some abbreviations. The symbols $\bar{\lambda}$, $\bar{x}$, and so on, abbreviate $(\lambda_1, \ldots, \lambda_k)$, $(x_1, \ldots, x_k)$, etc. Scalar multiplication is denoted by juxtaposition, thus

$$\bar{\lambda}\bar{x} = \sum_{i=1}^{k} \lambda_i x_i \qquad \text{and} \qquad \mu\bar{\lambda} = (\mu\lambda_1, \ldots, \mu\lambda_k)\,.$$

Finally, $\bar{0}$ stands for $(0, \ldots, 0)$.

Recall that an $\mathbf{R}$-module $\mathbf{M}$ is called **faithful** if $\lambda m = 0$ for all $m \in M$ implies that $\lambda = 0$. Suppose that the identity $f(\bar{x}) = g(\bar{x})$ is satisfied in a faithful module, where $f(\bar{x}) = \bar{\lambda}\bar{x}$ and $g(\bar{x}) = \bar{\mu}\bar{x}$. Then by substituting zeros we obtain that $\bar{\lambda} = \bar{\mu}$. Using this observation, the proof of the next lemma is an easy calculation, which is left to the reader.

LEMMA 5.5 *Let* $\mathbf{M}$ *be a faithful* $\mathbf{R}$*-module,*

$$\begin{aligned} f(x,\bar{u}) &= \lambda x + \bar{\mu}\bar{u} \in \mathrm{Clo}_{1+\mathrm{k}}\mathbf{M} \quad \textit{and} \\ p_i(x,y,z,\bar{u},\bar{v},w) &= \alpha_i x + \beta_i y + \gamma_i z + \bar{\tau}_i\bar{u} + \bar{\sigma}_i\bar{v} + \rho_i w \in \mathrm{Clo}_{4+2\mathrm{k}}\mathbf{M} \end{aligned}$$

for $0 \leq i \leq n$. *Then the identities in the previous lemma are satisfied if and only if the following equations hold in* $\mathbf{R}$ *for* $0 \leq i \leq n$*:*

$$\begin{array}{ll} \alpha_i = (\rho_{i+1} + \cdots + \rho_n)\lambda & (\textit{in particular} \quad \alpha_n = 0) \\ \beta_i = (\rho_0 + \cdots + \rho_{i-1})\lambda & (\textit{in particular} \quad \beta_0 = 0) \\ \gamma_i = \lambda & \\ \bar{\tau}_i = (\rho_{i+1} + \cdots + \rho_n)\bar{\mu} & (\textit{in particular} \quad \bar{\tau}_n = \bar{0}) \\ \bar{\sigma}_i = (\rho_0 + \cdots + \rho_{i-1})\bar{\mu} & (\textit{in particular} \quad \bar{\sigma}_0 = \bar{0}), \end{array}$$

and for $\rho = \rho_0 + \cdots + \rho_n$ *we have* $\rho\lambda = 0$ *and* $\rho\bar{\mu} = \bar{\mu}$.

Notice that the above lemma makes it possible to actually construct the terms $p_0, \ldots, p_n$ if $f(x,\bar{u})$ is given. We have to find elements $\rho_0, \ldots, \rho_n$ such that their sum, denoted by ρ, satisfies $\rho\lambda = 0$ and $\rho\bar{\mu} = \bar{\mu}$, and then define $p_0, \ldots, p_n$ by the equations above. Since our algebra $\mathbf{A}$ will be a reduct of a module, we have to investigate whether these newly constructed polynomials belong to particular subclones of Clo $\mathbf{M}$. We shall denote by $I(\bar{\eta})$ the ideal of $\mathbf{R}$ generated by $\eta_1, \ldots, \eta_k$ and for a term $f(\bar{x}) = \sum_{i=1}^{k} \eta_i x_i$ we put $I(f) = I(\bar{\eta})$. This definition is meaningful, since in a faithful module every term determines its coefficients.

LEMMA 5.6 *Let* $\mathbf{R}$, $\mathbf{M}$, f *be as in the previous lemma and* $\rho_0, \ldots, \rho_n$ *elements of* $\mathbf{R}$ *such that their sum, denoted by* ρ, *satisfies* $\rho\lambda = 0$ *and* $\rho\bar{\mu} = \bar{\mu}$. *Define* $p_0, \ldots, p_n$ *by the equations above.*

- **(i)** *For every* $0 \leq i \leq n$ *we have* $I(p_i) \subseteq I(\lambda, \bar{\mu}, \rho_i)$.
- **(ii)** *If* $\{\lambda, \rho_0, \ldots, \rho_n\}$ *is orthogonal and* $f \in \mathrm{Ort}(\mathbf{M})$, *then* $p_i \in \mathrm{Ort}(\mathbf{M})$ *for* $0 \leq i \leq n$.

Proof. The first statement is obvious. For the second, observe first that $\alpha_i = \beta_i = 0$, since $\{\lambda, \rho_0, \ldots, \rho_n\}$ is orthogonal. Now let this orthogonality be with respect to $\{\tilde{\lambda}, \tilde{\rho}_0, \ldots, \tilde{\rho}_n\}$, and let $\{\lambda, \bar{\mu}\}$ be orthogonal with respect to $\{\lambda^0, \bar{\mu}^0\}$. Then it is left to the reader to verify that

$$\{\gamma_i, \bar{\tau}_i, \bar{\sigma}_i, \rho_i\}$$

is orthogonal with respect to

$$\{\tilde{\lambda}\lambda^0, (\tilde{\rho}_{i+1} + \cdots + \tilde{\rho}_n)\bar{\mu}^0, (\tilde{\rho}_0 + \cdots + \tilde{\rho}_{i-1})\bar{\mu}^0, \tilde{\rho}_i\}\,.$$

Since zeros do not count in orthogonality, we indeed have $p_i \in \mathrm{Ort}(\mathbf{M})$ for $0 \leq i \leq n$. ∎

Before presenting our ring, we perform the analogous computation for the Hamiltonian property. Recall that a strongly Abelian variety is Hamiltonian iff for every term $f(x, \bar{u})$ there exists a binary term r such that the variety satisfies the identity $f(y, \bar{u}) = r(y, f(x, \bar{u}))$. The straightforward proof of the following lemma is also left to the reader.

LEMMA 5.7 *Let* $\mathbf{M}$ *be a faithful* $\mathbf{R}$*-module,*

$$\begin{aligned} f(x, \bar{u}) &= \lambda x + \bar{\mu}\bar{u} \in \mathrm{Clo}_{1+k}\mathbf{M} \quad \textit{and} \\ r(z, w) &= \gamma z + \rho w \in \mathrm{Clo}_2\mathbf{M}\,. \end{aligned}$$

Then the identity $f(y, \bar{u}) = r(y, f(x, \bar{u}))$ *is satisfied if and only if* $\gamma = \lambda$, $\rho\lambda = 0$, *and* $\rho\bar{\mu} = \bar{\mu}$. ∎

Consider a set $\{w, w_1, \ldots, w_n, \ldots\}$ of different symbols and let H be the set of all finite subsets of this set. Let ξ denote the singleton $\{w\}$ and for each $i \geq 1$, let ξ_i denote $\{w_i\}$. Then $\mathbf{H} = \langle H, \cup \rangle$ is a monoid, moreover, a semilattice. Let $\mathbf{S}$ be the ring of all formal linear combinations of the elements of H, with the coefficients taken from the ring of integers. Let $\mathbf{M} = {}_{\mathbf{S}}\mathbf{S}$ be the module obtained by the ring $\mathbf{S}$ considered as a left module over itself. Since $\mathbf{S}$ has an identity element, $\mathbf{M}$ is faithful.

Let $\mathcal{I}$ be the set of all ideals of $\mathbf{S}$ that do not contain ξ and let C be the clone $C = \mathrm{Ort}(M) \cap C_{\mathcal{I}}(\mathbf{M})$. Finally let $\mathbf{A}$ be the algebra with underlying set M and basic operations the elements of C.

THEOREM 5.8 *The variety generated by the algebra* $\mathbf{A}$ *defined above is strongly Abelian, but does not have the Hamiltonian property.*

Before commencing with the proof let us summarize some elementary properties of the ring $\mathbf{S}$. Recall that an element e of a ring is called idempotent if $e^2 = e$. It is obvious to see that a family $e_1, \ldots, e_k$ of idempotents is orthogonal iff $e_ie_j = 0$ for all $i \neq j$.

LEMMA 5.9 *Let* $\mathbf{S}$ *be the ring defined above.*

(i) *Let* $\phi : \{\xi, \xi_1, \dots, \xi_n, \dots\} \to \mathbf{S}$ *be any mapping such that* $\phi(\xi), \phi(\xi_i)$ *are all idempotent. Then* ϕ *can be extended to an endomorphism of* $\mathbf{S}$.

(ii) *Every element* η *of* $\mathbf{S}$ *can be written as* $n_1e_1+\cdots+n_ke_k$, *where* $e_1, \dots, e_k$ *is an orthogonal family of idempotents.*

(iii) *Let* $\eta = n_1e_1 + \cdots + n_ke_k$, *where* $e_1, \dots, e_k$ *is an orthogonal family of idempotents, and* $n_1, \dots, n_k \neq 0$. *Define* $e(\eta) = e_1+\cdots+e_k$. *Then* $e(\eta)$ *is idempotent, satisfying that* $e(\eta)\eta = \eta$ *and if* $\eta\theta = 0$ *then* $e(\eta)\theta = 0$ *and* $e(\eta)e(\theta) = 0$. *(Although not needed, it is not hard to prove that* $e(\eta)$ *is uniquely determined by* η.*)*

(iv) *A family* $\eta_1, \dots, \eta_k$ *in* $\mathbf{S}$ *is orthogonal iff* $\eta_i\eta_j = 0$ *for all* $i \neq j$.

PROOF. Since all elements of H can be written uniquely as products of elements of $\{\xi, \xi_1, \dots, \xi_n, \dots\}$, ϕ can be extended to a mapping from H to $\mathbf{S}$. This extension preserves multiplication, since the range of ϕ consists of idempotent elements. Now extend ϕ to $\mathbf{S}$ by linearity. It is an easy calculation to show that we get a ring–homomorphism.

Let $\eta = m_1h_1 + \cdots + m_\ell h_\ell$, where $h_1, \dots, h_\ell \in H$. We prove that η is a linear combination of orthogonal idempotents by induction on ℓ. For $\ell = 1$ the statement is trivial. Now suppose that $m_1h_1 + \cdots + m_{\ell-1}h_{\ell-1} = n_1e_1 + \cdots + n_ke_k$, where $e_1, \dots, e_k$ are orthogonal and idempotent. Then

$$\eta = \sum_{i=1}^{k} ((n_i + m_\ell)e_ih_\ell) + \sum_{i=1}^{k} (n_ie_i(1 - h_\ell)) + (m_\ell(1 - \sum_{i=1}^{k} e_i)h_\ell)\,,$$

and the $k + k + 1$ idempotents occurring in this decomposition are clearly orthogonal. Thus η is indeed a linear combination of orthogonal idempotents.

Let $\eta = n_1e_1 + \cdots + n_ke_k$ with $n_1, \dots, n_k \neq 0$, where $e_1, \dots, e_k$ is an orthogonal family of idempotents, and $e(\eta) = e_1 + \cdots + e_k$. Then it is clear that $e(\eta)$ is idempotent, and satisfies that $e(\eta)\eta = \eta$. Now let $\eta\theta = 0$. Then $0 = e_i\eta\theta = n_ie_i\theta$. As the additive group of $\mathbf{S}$ is torsion free, this implies $e_i\theta = 0$. Summing up we get $e(\eta)\theta = 0$. The same argument shows, that since $\theta e(\eta) = 0$, we get $e(\theta)e(\eta) = 0$.

Finally, let $\eta_1, \dots, \eta_k$ in $\mathbf{S}$ satisfy $\eta_i\eta_j = 0$ for all $i \neq j$. Then the family $\tilde{\eta}_i = e(\eta_i)$ shows that $\eta_1, \dots, \eta_k$ is orthogonal. ∎

Now we prove the Theorem. First we show that the variety generated by $\mathbf{A}$ is strongly Abelian. We apply Lemma 5.4 with $n = 2$. So let $f(x, \bar{u}) = \lambda x + \bar{\mu}\bar{u} \in \mathrm{Clo}_{1+\mathrm{k}}\mathbf{A}$ be given. If f is a projection, we can define p_0 and p_1 to be projections. Otherwise we have $\xi \notin I(\lambda, \bar{\mu})$. In order to construct the terms p_0 and p_1 in this case, we have to find elements of $\mathbf{S}$ such that the equations of Lemma 5.5 are satisfied. By the remark following this lemma, it is sufficient to find ρ_0 and ρ_1 in $\mathbf{S}$ such that their sum, denoted by ρ, satisfies $\rho\lambda = 0$ and $\rho\bar{\mu} = \bar{\mu}$. However, we have to ensure that our newly defined polynomials p_0 and p_1 are contained in the clone of $\mathbf{A}$. By Lemma 5.6 this will follow if $\xi \notin I(\lambda, \bar{\mu}, \rho_i)$ for $i = 0, 1$, and if $\{\lambda, \rho_0, \rho_1\}$ is an orthogonal family.

Define $\rho = e(\mu_1) + \cdots + e(\mu_k)$. Since $\bar{\mu}$ is orthogonal, Lemma 5.9 shows that ρ is an idempotent, $\rho\lambda = 0$ and $\rho\bar{\mu} = \bar{\mu}$ is satisfied. Consider the elements $\xi, \rho, \lambda, \bar{\mu}$ of $\mathbf{S}$. These can all be expressed by using finitely many of the generators, say ξ and $\xi_1, \ldots, \xi_n$, of $\mathbf{S}$. Let $x = \xi_{n+1}$ and define $\rho_0 = \rho x$ and $\rho_1 = \rho(1 - x)$. We show that these elements satisfy the above conditions.

It is clear that $\{\lambda, \rho_0, \rho_1\}$ is an orthogonal family. Suppose, to get a contradiction, that $\xi \in I(\lambda, \bar{\mu}, \rho_0)$, that is, ξ is a linear combination of these elements using coefficients from $\mathbf{S}$. Construct an endomorphism ϕ of $\mathbf{S}$ that fixes ξ and $\xi_1, \ldots, \xi_n$, but maps x to 0. This will take this linear combination into another one which shows that $\xi \in I(\lambda, \bar{\mu})$, which contradicts f being in the clone of $\mathbf{A}$. To show that $\xi \notin I(\lambda, \bar{\mu}, \rho_1)$ construct another ϕ, which maps x to 1. Thus we have proved that $\mathbf{A}$ indeed generates a strongly Abelian variety.

Finally suppose that this variety is Hamiltonian. Define the operation $f(x, u) = (2 - 2\xi)x + 2\xi u$. Then f is in the clone of $\mathbf{A}$, since the ideal $2\mathbf{S}$ of $\mathbf{S}$ does not contain ξ. Applying Lemma 5.7 we obtain that the clone of $\mathbf{A}$ contains a binary operation $r(z, w) = (2-2\xi)z + \rho w$, where $\rho(2-2\xi) = 0$ and $\rho 2\xi = 2\xi$. Adding up the last two equalities we get that $2\rho = 2\xi$, hence $\rho = \xi$. This is a contradiction, since r is not a projection, and the ideal generated by its coefficients does not contain ξ. Thus the proof of th e theorem is complete. ∎

6 A non quasi-affine example

We present an example of a five element algebra $\mathbf{A}$ that generates a strongly Abelian variety, but which is not quasi-affine. Hence the structure theorem of Shapiro [12] of strongly Abelian varieties with one basic operation cannot be generalized in a straightforward manner.

Quasi affine algebras have been characterized by Quackenbush [11] via a series of implications (universal Horn formulae). We shall exhibit the failure in $\mathbf{A}$ of one of these implications.

Definition 6.1 An algebra is said to satisfy the *Two Terms Condition* (TTC) if for every two terms $f(\bar{x},\bar{y})$ and $g(\bar{x},\bar{y})$ and elements $\bar{a}$, $\bar{a}'$, $\bar{b}$, $\bar{b}'$, $\bar{c}$, $\bar{c}'$, $\bar{d}$, and $\bar{d}'$, the first three of the equations

$$\begin{aligned} f(\bar{a},\bar{b}) &= g(\bar{c},\bar{d}) \\ f(\bar{a}',\bar{b}) &= g(\bar{c}',\bar{d}) \\ f(\bar{a}',\bar{b}') &= g(\bar{c}',\bar{d}') \\ f(\bar{a},\bar{b}') &= g(\bar{c},\bar{d}') \end{aligned}$$

implies the fourth one.

It is a very easy exercise for the reader to show that quasi-affine algebras satisfy the two terms condition. To define the algebra $\mathbf{A}$, consider the elements $p=(0,0)$, $q=(0,1)$, $r=(1,0)$, $s=(1,1)$, and define multiplication by $(x,y)*(u,v)=(x,v)$. This algebra is a well-known semigroup, we denote it by $\mathbf{S}$. Consider now two copies of $\mathbf{S}$, on the sets $\{p,q,r,s_1\}$ and $\{p,q,r,s_2\}$, respectively, with operations $*_1$ and $*_2$. The underlying set of $\mathbf{A}$ is defined to be $A=\{p,q,r,s_1,s_2\}$. Define the mappings e_i $(i=1,2)$ to be the identity map on $\{p,q,r\}$ and let $e_1(s_1)=e_1(s_2)=s_1$, and $e_2(s_1)=e_2(s_2)=s_2$. The binary operation $e_i(u)*_i e_i(v)$ on $\{p,q,r,s_1,s_2\}$ is clearly an extension of $*_i$, let us denote it also by $*_i$. Finally, let the basic operations of $\mathbf{A}$ be $*_1$ and $*_2$.

To show that $\mathbf{A}$ fails the two terms condition observe that

$$\begin{aligned} r *_1 p &= r = & r *_2 p \\ p *_1 p &= p = & p *_2 p \\ p *_1 q &= q = & p *_2 q\,, \end{aligned}$$

but $r *_1 q = s_1 \neq s_2 = r *_2 q$.

To prove that $\mathbf{A}$ generates a strongly Abelian variety we will use the characterization given in Corollary 3.6. One can check that for any i and $j \leq 2$, $\mathbf{A}$ satisfies the identities

$$(x_1 *_i x_2) *_j x_3 \approx x_1 *_j x_3$$

and

$$x_1 *_i (x_2 *_j x_3) \approx x_1 *_i x_3.$$

From this it follows that every term operation of $\mathbf{A}$ is essentially equal to either a projection or one of the operations $x *_1 x$, $x *_2 x$, $x *_1 y$ or $x *_2 y$.

Thus by Corollary 3.6 it will suffice to find terms $l_i(x,y)$ and $r_i(x,y)$ for $i \leq 2$ such that

$$\mathbf{A} \models l_i(x *_i y, z) \approx z *_i y$$

and

$$\mathbf{A} \models r_i(x *_i y, z) \approx x *_i z.$$

Setting $l_i(x,y) = y *_i x$ and $r_i(x,y) = x *_i y$ works.

An interesting feature of the algebra $\mathbf{A}$ is that its clone is the set-theoretic union of the clones of the algebras $\langle A, *_1 \rangle$ and $\langle A, *_2 \rangle$. This follows from the preceding discussion.

7 Conclusion

We have seen that locally finite strongly Abelian varieties are Hamiltonian and in some sense can be regarded as generalized unary algebras. Under further assumptions it can be established that these varieties are actually equivalent to varieties of multi-sorted unary algebras. If the variety is assumed have a decidable theory ([9]), or if it fails to have the maximum number of nonisomorphic models in some infinite cardinality ([3]) or if the isomorphism problem for the variety is computable in polynomial time ([15]) then such a "nice" structure theory is obtained.

The question of whether or not all locally finite Abelian varieties are Hamiltonian is still open, but recently the second author has made some progress towards settling this problem ([14]). He has shown that every finite simple Abelian algebra is Hamiltonian i.e., such an algebra has no nontrivial subalgebras. In [7] McKenzie has proved some interesting results in this area.

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