ON THE LOEWY RANK OF INFINITE ALGEBRAS

Emil W. Kiss

The Loewy rank of a complete lattice L is defined as follows. Take the meet a_1 of all coatoms of L. Then let a_2 be the meet of all lower covers of a_1. Iterate this process to define a_α for every ordinal α by letting $a_{\alpha+1}$ be the meet of all lower covers of a_α, and a_α the meet of all a_β (for $\beta < \alpha$) if α is a limit ordinal. The Loewy rank of L is the smallest ordinal α for which a_α is the zero of L, and the symbol ∞ if such α does not exist. The Loewy rank $\mathbf{L}(A)$ of an algebra A is defined to be the Loewy rank of $\mathrm{Con}(A)$, and for a class $\mathcal{K}$ of algebras $\mathbf{L}(\mathcal{K}) = \sup\{\mathbf{L}(A) \mid A \in \mathcal{K}\}$.

The above concept has been introduced in 2.2 of Ralph McKenzie's paper [3]. There he proves (in Corollary 2.11) that if A is a finite algebra generating a congruence modular variety $\mathcal{V}$, then the Loewy rank of any finite member of $\mathcal{V}$ is bounded by $\mathbf{L}(\mathbf{S}(A))$. This statement follows from two easy observations. The first one is that the Loewy rank of a subdirect product (in a modular variety) cannot exceed the supremum of the Loewy ranks of its factors. The second observation is that $\mathbf{L}(\mathbf{H}(A)) \le \mathbf{L}(A)$ provided that $\mathrm{Con}(A)$ is finite and modular. This second statement does not hold for infinite lattices, however (take the ring of integers). Therefore the above argument does not give a bound for the infinite members of $\mathcal{V}$. In 2.13 and 3.11 of [3] Ralph McKenzie asks whether $\mathbf{L}(\mathcal{V})$ is finite. We answer this question by proving

Theorem. *Let A be a finite algebra in a modular variety. Then $\mathbf{L}(\mathbf{V}(A)) \le |A|$.*

We shall prove a better result actually, but we were not able to show that $\mathbf{L}(\mathcal{V}) = \mathbf{L}(\mathcal{V}_{\mathrm{fin}})$ for every finitely generated modular variety $\mathcal{V}$. In fact, the proof below indicates that this might not be the case in general (see the discussion at the end).

The idea of the proof is to extend a group theoretic lemma, found in L. G. Kovács, M. F. Newman [2], to the congruence modular case, and to infinite subdirectly irreducibles. In group theory, subdirectly irreducible algebras are sometimes called monolithic. The monolith of a subdirectly irreducible algebra S, that is, its smallest nonzero congruence, is denoted by $\mu(S)$ in this paper. Let us call a class $\mathcal{S}$ of (subdirectly irreducible) algebras *closed under monolithic sections* if $\mathbf{SiHS}(\mathcal{S}) = \mathcal{S}$.

Lemma. *Let $\mathcal{S}$ be a finite set of finite subdirectly irreducible algebras closed under monolithic sections such that $\mathcal{V} = \mathbf{V}(\mathcal{S})$ is congruence modular. If S is a subdirectly irreducible algebra in $\mathcal{V}$, then*

$$S/\mu(S) \in \mathbf{V}\{D/\mu(D) \mid D \in \mathcal{S}\}\,.$$

Proof. First we prove this statement for finite S. Apply the proof of Theorem 10.12 in the book [1] of R. Freese and R. Mckenzie. We obtain a finite subdirect product

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

C of subdirectly irreducible algebras $A_0, \dots, A_{k-1} \in \mathcal{S}$ such that $S \cong C/\vartheta$ for some congruence ϑ of C, and such that the projection kernels $\eta_0, \dots, \eta_{k-1}$ and the congruences defined by ${\eta_j}' = \prod\{\eta_i \mid i \neq j\}$ satisfy

$$\vartheta^* = \vartheta + \vartheta^* {\eta_i}' \qquad \text{and} \qquad {\eta_i}^* = \eta_i + \vartheta^* {\eta_i}'$$

for all i. Here, α^* is the unique cover of a congruence $\alpha \in \mathrm{Con}(A)$, that is, $\alpha^*/\alpha = \mu(A/\alpha)$. We prove by induction on i that

$${\eta_0}^* \dots {\eta_i}^* = \eta_0 \dots \eta_i + \vartheta^* {\eta_0}' + \dots + \vartheta^* {\eta_i}' .$$

Indeed, this is clear for $i = 0$. On the other hand, if it holds for $i - 1$, then by modularity,

$$\begin{aligned} &{\eta_0}^* \dots {\eta_i}^* = \\ &= (\eta_0 \dots \eta_{i-1} + \vartheta^* {\eta_0}' + \dots + \vartheta^* {\eta_{i-1}}') {\eta_i}^* = \eta_0 \dots \eta_{i-1} {\eta_i}^* + \vartheta^* {\eta_0}' + \dots + \vartheta^* {\eta_{i-1}}' , \end{aligned}$$

and by modularity again,

$$\eta_0 \dots \eta_{i-1} {\eta_i}^* = \eta_0 \dots \eta_{i-1} (\eta_i + \vartheta^* {\eta_i}') = \eta_0 \dots \eta_i + \vartheta^* {\eta_i}' ,$$

proving our assertion. But we have $\eta_0 \dots \eta_{k-1} = 0$, hence

$${\eta_0}^* \dots {\eta_{k-1}}^* \leq \vartheta^* .$$

This yields a subdirect decomposition of a coimage of $C/\vartheta^* \cong S/\mu(S)$ with components $C/{\eta_i}^* \cong A_i/\mu(A_i)$, proving the Lemma for finite S.

For infinite S we use the idea of proving Quackenbush's Theorem [4]. However, modularity is essential in the following argument.

Claim. *Let S be a subdirectly irreducible algebra in a finitely generated congruence modular variety. Then*

$$S/\mu(S) \in \mathbf{V}\{D/\mu(D) \mid D \in \mathbf{V}(S), D \text{ finite and subdirectly irreducible}\} .$$

Proof. If $\mu(S)$ is not Abelian, then S is finite by Jónsson's Lemma, and we are done. By Theorem 10.16 of [1], the size of the blocks of $\mu(S)$ is bounded by a finite number n. Let $\mu(S) = \mathrm{Cg}_S(a, b)$. It is sufficient to prove that for every finite $B' \leq S/\mu(S)$ containing $a/\mu(S)$ there exists a finite, subdirectly irreducible $D \in \mathbf{V}(S)$ with $B' \in \mathbf{HS}(D/\mu(D))$.

Let B be the coimage of B' in S. Then B is finite, since it is the union of finitely many blocks of $\mu(S)$. For every $c \neq d \in B$ we have $(a, b) \in \mathrm{Cg}_S(c, d)$. By Mal'cev's Lemma, this congruence spreads in a finitely generated subalgebra of S. Doing this for all pairs in B^2, we obtain a finitely generated, hence finite subalgebra C of S such that $B \leq C$, and $c \neq d$, $c, d \in B$ imply that $(a, b) \in \mathrm{Cg}_C(c, d)$.

Let ψ be a maximal congruence of C with respect to $(a,b)\notin\psi$. Then $D=C/\psi$ is subdirectly irreducible with monolith ψ^*/ψ, where $\psi^*=\psi+\mathrm{Cg}_C(a,b)$. We show that

$$(*)\qquad \mu(S)\restriction B\supseteq\psi^*\restriction B\,.$$

Indeed, let $(c,d)\in\psi^*\restriction B$. As $\mu(S)$ is Abelian, so is $\mathrm{Cg}_C(a,b)\subseteq\mathrm{Cg}_S(a,b)=\mu(S)$, hence it permutes with ψ by H. P. Gumm's result (see Theorem 6.2 of [1]). Thus, there exists $e\in C$ with $c\psi e\mathrm{Cg}_C(a,b)d$. But B is a union of $\mu(S)$-blocks, hence it is a union of $\mathrm{Cg}_C(a,b)$-blocks, hence $e\in B$. By the property of C above, if $e\neq c$, then $(a,b)\in\mathrm{Cg}_C(e,c)$, so $(a,b)\in\psi$, a contradiction. Therefore $e=c$, hence $(c,d)\in\mathrm{Cg}_C(a,b)\subseteq\mu(S)$, proving (*).

To conclude the proof of the Claim (and hence of the Lemma), observe that $B'\cong B/(\mu(S)\restriction B)\in\mathbf{H}(B/(\psi^*\restriction B))$, and $B/(\psi^*\restriction B)\leq C/\psi^*\cong D/\mu(D)$.

To prove the Theorem, let $\mathcal{K}$ be a finite set of finite algebras generating a modular variety $\mathcal{V}$, set $\mathcal{S}=\mathbf{SiHS}(\mathcal{K})$ and $s(\mathcal{K})=\max\{|S|\mid S\in\mathcal{S}\}$. It is sufficient to show that $\mathbf{L}(\mathcal{V})\leq s(\mathcal{K})$, consider a failure with $s(\mathcal{K})$ being minimal. Let $\mathcal{V}'$ be generated by $\mathcal{K}'=\{D/\mu(D)\mid D\in\mathcal{S}\}$. Clearly, $s(\mathcal{K}')<s(\mathcal{K})$, and the Lemma shows that $\mathbf{L}(\mathcal{V})\leq\mathbf{L}(\mathcal{V}')+1$, concluding the proof.

Remarks. It would be interesting to know whether $\mathbf{L}(\mathcal{V})$ can exceed $\mathbf{L}(\mathcal{V}_{\mathrm{fin}})$ for finitely generated modular varieties. We know that $\mathbf{L}(\mathbf{V}(\mathcal{K})_{\mathrm{fin}}))=\mathbf{L}(\mathbf{S}(\mathcal{K}))$ holds. The above proof yields a sharper bound on $\mathbf{L}(\mathbf{V}(\mathcal{K}))$ than $\max\{|S|\mid S\in\mathcal{K}\}$. Indeed, iterate the construction of $\mathcal{K}'$ from $\mathcal{K}$ given in the previous paragraph, and let $\ell(\mathcal{K})$ be the number of steps in which $\mathcal{K}$ evaporates. Then we see that $\mathbf{L}(\mathbf{V}(\mathcal{K}))\leq\ell(\mathcal{K})$. It is not necessary though that $\ell(\mathcal{K})\leq\mathbf{L}(\mathbf{S}(\mathcal{K}))$. Indeed, if $\mathcal{K}$ consists of a single subdirectly irreducible algebra A, and A has a subalgebra B such that $\mu(A)\restriction B=0_B$, but B has a large Loewy rank, more precisely, $\mathbf{L}(\mathbf{S}(A))=\mathbf{L}(\mathbf{S}(B))$, then forming $\mathcal{K}'$ the Loewy rank remains the same. This situation is easy to model, moreover, according to P. P. Pálfy, the group $A=SL_2(163)$ has such a subgroup B (with Loewy rank 5).

What seems much harder to do (if at all possible), is to find A and B so that an infinite subdirectly irreducible algebra $S\in\mathbf{V}(A)$ can also be constructed with $\mathbf{L}(S/\mu(S))=\mathbf{L}(B)$. This might be accomplished by analysing the proof of the Lemma. However, it seems certain that the infinite subdirectly irreducibles can be nastier than the finite ones. Using a more explicit form of the Freese-McKenzie construction of subdirectly irreducibles, it seems to me that an infinite $S\in\mathbf{V}(A)$ can be constructed with $B\in\mathbf{H}(S/\mu(S))$ (which is, of course, still impossible for finite S). It would also be very interesting to investigate further the situation in the proof of Theorem 10.12 of [1], for example to find canonical frames. This could lead to a better understanding of subdirectly irreducibles in finitely generated modular varieties.

Acknowledgements. This proof came up during a conversation with L. G. Kovács at the Australian National University in Canberra. It was Laci Kovács who called my attention to the group theoretic lemma due to him and M. F. Newman that has been generalized above. He also found a solution to McKenzie's problem for the case of

groups. His argument is completely different from mine, and uses the Oates-Powell theorem. I am very grateful to him for inviting me to Canberra, and for his patience in answering my many questions on groups.

References

1. R. Freese, R. McKenzie, *Commutator theory for congruence modular varieties.* London Math. Soc. Lecture Notes Series, vol. 125, Cambridge University Press, 1987.
2. L. G. Kovács, M. F. Newman, *On critical groups*, J. Austral. Math. Soc. (1966), 237-250.
3. R. McKenzie, *Nilpotent and solvable radicals in locally finite congruence modular varieties*, preprint.
4. R. W. Quackenbush, *Equational classes generated by finite algebras*, Algebra Universalis **1** (1971), 265-266.

Mathematical Institute of the Hung. Acad. of Sci., 1364. Budapest, P. O. Box 127, Hungary