AN EASY WAY TO MINIMAL ALGEBRAS

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Dedicated to the memory of Alan Day

Abstract. A finite algebra **C** is called minimal with respect to a pair $\delta < \theta$ of its congruences if every unary polynomial f of **C** is either a permutation, or $f(\theta) \subseteq \delta$. It is the basic idea of tame congruence theory developed by Ralph McKenzie and David Hobby [7] to describe finite algebras via minimal algebras that sit inside them. As shown in [7], minimal algebras have a very restricted structure.

This paper presents a new tool, the Twin Lemma, which makes it possible to give short proofs of some of these structure theorems. This part can be read as an alternative introduction to the theory. Our method yields new information in the type **1** case, and is especially useful in describing E-minimal algebras (that is, algebras that are minimal with respect to every prime congruence quotient). We complete their theory given in [7] by proving a structure theorem for the type **1** case. Finally we show that if an algebra is minimal with respect to two quotients, then the two types are the same, and if this type is **2**, **3**, or **4**, then the bodies are also equal.

1. Introduction

Tame congruence theory is a powerful, deep branch of universal algebra, which has allowed researchers to approach and solve problems that seemed hopelessly difficult before, and which could be applied fruitfully in other research areas as well. The main tools are presented in the book of Ralph McKenzie and David Hobby [7], but the theory is not so easy to learn. Since our paper investigates one of these tools (namely the structure of $\langle \delta, \theta \rangle$-minimal algebras), we have elected to write the corresponding Section 3 so as to be accessible for beginners in this area. The reader (having a background in universal algebra) can read Section 2, and some proofs referred to in that section, to learn the necessary preliminaries, and then go to Section 3, which will help him understand the structure of minimal algebras. Our route yields an alternative approach to the results of Chapter 4 in [7]. We suggest that the reader consult the book [7] continuously while reading this paper.

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The paper also offers new results. The Twin Lemma 3.1 gives new information for the unary type (see also Theorem 3.4). This helps in characterizing the clones of E-minimal algebras of type **1**, thus supplementing the results of Chapter 13 of [7]. This characterization is given in Theorem 4.4. Finally our result on algebras that are minimal with respect to more than one quotient is given in Theorems 5.2 and 5.1.

The results presented here have applications in the investigation of residually small varieties (see [11]), and also in the investigation of minimal sets in subdirect powers (see [10]). We call the reader's attention to the paper [9] of Keith Kearnes, which proves deep results on centrality, and some ideas of which are related to the ones presented here. We think that the present paper is good preliminary reading for [9].

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2. Basic concepts and results

The notation used in the paper is mostly the same as that used in [7]. In particular, algebras are denoted by boldface capital letters, and A is the underlying set of $\mathbf{A}$. Boldface lower case letters, like $\mathbf{b}$, denote elements of cartesian products of sets, and b_i stands for the i-th component of $\mathbf{b}$. We start counting the elements from 1, so $\mathbf{b}$ typically denotes $(b_1, \ldots, b_n)$ for some integer n. Many times, in particular, when considering polynomials of products of algebras, it is much easier to understand definitions or arguments if elements of these products are written as *column vectors,* while multiple arguments of functions are written as row vectors. If R is a binary relation, then by $\mathbf{a}\, R\, \mathbf{b}$ we mean $a_i \; R \; b_i$ for all i.

Commutator theory, originated by J. Smith [12], and developed by C. Herrmann, W. Taylor, R. McKenzie, R. Freese, H. P. Gumm, and others, was a great discovery in the period of 1976–80. Its significance can be compared to that of tame congruence theory. The two main references are [5] and [2], or the reader may look at the paper [4] for a quick introduction. Here we summarize only the facts necessary to understand our paper.

Commutator theory works for varieties of algebras whose congruence lattices are modular. The basic idea is that for such varieties one can give a common generalization of the concept of the commutator subgroup $[N, M]$ of two normal subgroups of a group, and of the product $IJ + JI$ of two ideals of a ring. The modular commutator is a binary operation on the congruence lattice of algebras, which has useful algebraic properties, but, more importantly, gives a very good structure theory when the commutator is 'small' (as abelian groups and rings with zero multiplication are well-behaved with respect to general groups and general rings). These properties are summarized in the following theorem. Recall that for a congruence α of an algebra $\mathbf{A}$ and mapping $\phi : \mathbf{A} \to \mathbf{B}$ we define $\phi(\alpha)$ to be the congruence of $\mathbf{B}$ generated by all pairs $(\phi(a), \phi(b))$ with $a\ \alpha\ b$.

Theorem 2.1. *There exists a binary operation* $[\ ,\ ]$ *on the congruence lattice of every algebra* $\mathbf{A}$ *belonging to a congruence modular variety such that for all congruences* α, β, β_i *of* $\mathbf{A}$ *we have:*

(1) $[\alpha, \beta] = [\beta, \alpha] \leq \alpha \wedge \beta$.
(2) $[\alpha, \bigvee \beta_i] = \bigvee [\alpha, \beta_i]$.
(3) *If* $\varphi : \mathbf{A} \to \mathbf{B}$ *is an epimorphism, then* $[\varphi(\alpha), \varphi(\beta)] = \varphi[\alpha, \beta]$.
(4) *If* $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$ *(the algebra* $\mathbf{A}$ *is abelian), then* $\mathbf{A}$ *is polynomially equivalent to a module, that is,*
    (a) *an abelian group addition* $+$ *can be defined on the underlying set of* $\mathbf{A}$ *such that all basic operations of* $\mathbf{A}$ *are of the form*

$$f(x_1, \dots, x_n) = r_1 x_1 + \cdots + r_n x_n + c\,,$$

    where the r_i *are abelian group endomorphisms, and* $c \in A$*;*
    (b) $x - y + z$ *is a term function of* $\mathbf{A}$.

Algebras satisfying conditions (a) and (b) in statement (4) are called *affine*. Once the commutator is defined, we can speak of abelian (solvable, nilpotent) algebras or congruences (as their definitions from group theory carry over without modification).

There are several ways to actually define the commutator operation itself, but these are not easy to comprehend, and for a newcomer it is probably easier to work with the statements above, considered as axioms. We shall reproduce two of the equivalent definitions. Both of these definitions are valid for arbitrary algebras (although modularity is essential in proving Theorem 2.1). Lemmas 2.3 and 2.5 also hold without assuming modularity, and we strongly recommend that the reader prove them as exercises.

Definition 2.2. Let $\mathbf{A}$ be an algebra, L and R binary relations on A, and δ a congruence of $\mathbf{A}$. Two polynomials $g(\mathbf{x})$ and $h(\mathbf{x})$ of $\mathbf{A}$ are called *R-twins*, if they are of the form $g(\mathbf{x}) = f(\mathbf{x}, \mathbf{c})$ and $h(\mathbf{x}) = f(\mathbf{x}, \mathbf{d})$ for some polynomial f of $\mathbf{A}$ and vectors $\mathbf{c}\ R\ \mathbf{d}$ of A (of appropriate length).

We say that L *centralizes* R modulo δ, or that the *$\langle L, R\rangle$-term condition* holds modulo δ (in notation: $C(L, R; \delta)$) if for all polynomials f of $\mathbf{A}$, elements $a\ L\ b$, and vectors $\mathbf{c}\ R\ \mathbf{d}$ of A,

$$\begin{array}{ccc} f(a, \mathbf{c}) & \delta & f(a, \mathbf{d}) \\ & \Updownarrow & \\ f(b, \mathbf{c}) & \delta & f(b, \mathbf{d})\,. \end{array}$$

The *commutator* $[L, R]$ of L and R is defined to be the smallest congruence δ of $\mathbf{A}$ with $C(L, R; \delta)$. The largest congruence α of $\mathbf{A}$ satisfying $C(\alpha, R; \delta)$ is denoted by $(\delta : R)$. We write $\operatorname{ann}(R)$ for $(0_A : R)$; this is the *annihilator* of R.

Thus, two polynomials are R-twins if they can be derived from the same polynomial with different parameter sequences which are R-related componentwise, and the $\langle L, R\rangle$-term condition says that if two unary R-twins agree modulo δ at a, then they also agree at all other places that are L-related to a.

Note that if $\overline{R}$ denotes the compatible tolerance of $\mathbf{A}$ generated by R, then $C(L, R; \delta)$ is equivalent to $C(L, \overline{R}; \delta)$. Thus it is usual to assume that R is reflexive and symmetric. Similarly, it is sufficient to consider those relations L that are actually congruences, as shown by the following lemma.

Lemma 2.3. *Assume the notation in Definition 2.2.*

(1) *The set of all congruences δ satisfying $C(L, R; \delta)$ is closed under intersection (so the commutator $[L, R]$ indeed exists).*
(2) *For groups and rings the commutator coincides with the concepts mentioned above.*
(3) *The set of all pairs (a, b) (not necessarily in L) satisfying the condition in Definition 2.2 forms a congruence relation of $\mathbf{A}$. This congruence is $(\delta : R)$ (so the definition of $(\delta : R)$ is meaningful).*
(4) *By the existence of $(\delta : R)$ we have*

$$C(\bigvee_{i\in I} \alpha_i, R; \delta) \quad \Longleftrightarrow \quad (\forall i \in I)C(\alpha_i, R; \delta),$$

where $\alpha_i \in \operatorname{Con}(\mathbf{A})$ for $i \in I$.

The significance of twins is partially explained by the following fact. Consider a compatible binary relation R of $\mathbf{A}$ as a subalgebra of $\mathbf{A} \times \mathbf{A}$. Then the polynomials of this subalgebra are exactly the pairs of R-twins of $\mathbf{A}$ acting componentwise. This observation is the basis of an alternative, 'semantic' definition of the commutator, given by H. P. Gumm in [5].

Definition 2.4. If $\mathbf{A}$ is an algebra and R is a compatible, reflexive, binary relation on $\mathbf{A}$, then the subalgebra of $\mathbf{A}^2$ with underlying set R (that is, all R-related pairs) is sometimes (particularly, when R is a congruence β) denoted by $\mathbf{A}(R)$ (instead of $\mathbf{R}$). If L is any binary relation on A, then $\Delta_{L,R}$ denotes the congruence on $\mathbf{A}(R)$ generated by

$$\{\langle (x, x), (y, y)\rangle \,:\, x\ L\ y\}\,.$$

Lemma 2.5. *We have $[L, R] = 0$ if and only if the diagonal subuniverse of $\mathbf{A}(R)$ (that is, the set $\{(x, x) : x \in A\}$) is a union of $\Delta_{L,R}$-classes.*

The commutator can be defined using this observation by considering factor algebras. In the paper [5], Theorem 2.1 is proved using this semantic definition. The nice pictures of the congruence-class geometries presented in this paper are really helpful in understanding the proofs. We prefer the TC-commutator (defined in Definition 2.2) in two respects.

First, it yields a reasonably simple proof of the fundamental theorem of abelian algebras (statement (4) of Theorem 2.1, originally proved by C. Herrmann [6]). The reader is encouraged to prove the following statement due to J. Smith [12] and H. P. Gumm [3] (for hints see Exercise 3.2 (3) of [7]). To lift this result to the general modular case, it may be worth reading W. Taylor's proof in [13].

Lemma 2.6. *If* $\mathbf{A}$ *is an abelian algebra in a congruence permutable variety, then* $\mathbf{A}$ *is affine.*

The second reason to use the TC-commutator is that it is better suited to go beyond congruence modularity. Of course, many good properties listed in Theorem 2.1 are lost. Homomorphic images of abelian algebras are not necessarily abelian, the commutator is not necessarily commutative, and, despite Lemma 2.3 (4), it is not necessarily distributive over joins (not even in the first variable). It remains, however, a useful tool in tame congruence theory.

Since the commutator is not commutative in general, it is important to remember, which side is left and which side is right in Definition 2.2. Recall that the $\langle L, R\rangle$-term condition says that if two R-twins (R is the relation on the right) agree modulo δ at a, then they also agree at all other places that are L-related to a. Thus we move around in L (the relation on the left) with the variable of the twin polynomials. So it makes no difference if a and b are vectors or single elements, since we can move the components one at a time. On the other hand, it is important to assume that $\mathbf{c}$ and $\mathbf{d}$ are vectors. If these are single elements, we get the *binary term condition.*

Definition 2.7. Let $\mathbf{A}$ be an algebra, L and R binary relations on A, and δ a congruence of $\mathbf{A}$. We say that the $\langle L, R\rangle$-*binary term condition* holds modulo δ (in notation: $C^2(L, R; \delta)$) if for all binary polynomials f of $\mathbf{A}$ and elements $a\ L\ b$ and $c\ R\ d$ of A,

$$\begin{array}{ccc} f(a,c) & \delta & f(a,d) \\ & \Updownarrow & \\ f(b,c) & \delta & f(b,d)\,. \end{array}$$

Earlier the name 'weak term condition' has been used for this concept, but this name has a different (although related) meaning in [9]. In the modular case, the binary term condition is equivalent to the term condition. The reader may try to prove that an algebra having a Mal'cev term, and satisfying $C^2(1, 1; 0)$ is affine (see the proof of Theorem 4.7 in [7]). This statement is a consequence of the following trick, which is hidden in Chapter 4 of [7].

Lemma 2.8. *If an algebra* $\mathbf{A}$ *admits a Mal'cev polynomial, then*

$$C^2(L, R; \delta) \Longrightarrow C(R, L; \delta) \Longrightarrow C(L, R; \delta)$$

for all binary relations L, R *of* $\mathbf{A}$ *and* $\delta \in \operatorname{Con}(\mathbf{A})$.

Proof. It is sufficient to prove the first implication, it obviously implies the second one (by reversing the roles of L and R). So assume that $a\ R\ b$ and $\mathbf{c}\ L\ \mathbf{d}$ are elements and vectors in A, and $f(a,\mathbf{c})\ \delta\ f(a,\mathbf{d})$. For a Mal'cev polynomial m set

$$g(x,\mathbf{z}) \stackrel{\text{def}}{=} m(f(x,\mathbf{z}), f(x,\mathbf{d}), f(b,\mathbf{d}))\,.$$

Then $g(b,\mathbf{z}) = f(b,\mathbf{z})$, $g(a,\mathbf{d}) = f(b,\mathbf{d})$, and $g(a,\mathbf{c})\ \delta\ f(b,\mathbf{d})$ since $f(a,\mathbf{c})\ \delta\ f(a,\mathbf{d})$. Thus we have

$$\begin{array}{ccc} g(a,\mathbf{c}) & \delta & g(a,\mathbf{d}) \\ & & \| \\ g(b,\mathbf{c}) & & g(b,\mathbf{d}) \\ \| & & \| \\ f(b,\mathbf{c}) & & f(b,\mathbf{d})\,. \end{array}$$

Applying the binary term condition, from $g(a,\mathbf{d})\ \delta\ g(b,\mathbf{d})$ we get $g(a,\mathbf{c})\ \delta\ g(b,\mathbf{c})$ (by switching the components of $\mathbf{c}$ one at a time to the components of $\mathbf{d}$). Thus transitivity implies $g(b,\mathbf{c})\ \delta\ g(b,\mathbf{d})$. Hence $f(b,\mathbf{c})\ \delta\ f(b,\mathbf{d})$, as desired. ☐

The statement of this lemma actually holds under slightly weaker conditions, with the same proof. Namely, m has to be Mal'cev only on the range of f. We record the exact statement here for future reference.

Lemma 2.9. *Let* $\mathbf{A}$ *be an algebra,* L *and* R *binary relations of* $\mathbf{A}$*, and* $\alpha \leq \delta$ *congruences of* $\mathbf{A}$ *such that* $\mathbf{A}$ *satisfies* $C^2(L,R;\alpha)$*. Let* f *be a polynomial,* $a\ R\ b$ *and* $\mathbf{c}\ L\ \mathbf{d}$ *elements and vectors of* $\mathbf{A}$*, and* $s = f(a,\mathbf{c})$*,* $t = f(a,\mathbf{d})$*,* $u = f(b,\mathbf{c})$*,* $v = f(b,\mathbf{d})$*. Suppose that* $\mathbf{A}$ *has a ternary polynomial* d *satisfying* $d(u,v,v) = u$*,* $d(v,v,v) = v$*, and* $d(t,t,v) = v$*. Then*

$$\begin{array}{ccc} s = f(a,\mathbf{c}) & \delta & t = f(a,\mathbf{d}) \\ & \Downarrow & \\ u = f(b,\mathbf{c}) & \delta & v = f(b,\mathbf{d})\,. \end{array}$$

As the commutator is now noncommutative, the definition of nilpotence must be adjusted (solvability remains the same). We say that an algebra $\mathbf{A}$ is left nilpotent if

$$[1_A,[1_A,[1_A,\ldots,[1_A,1_A]\ldots]]] = 0_A$$

(for a sufficiently long expression). Right nilpotence is defined analogously. One of the main results of K. Kearnes [9] states that for finite algebras left nilpotence is the weakest form of nilpotence, implied, for example, by right nilpotence.

We should really speak of a left annihilator in Definition 2.2, but we shall not, because there is no such natural definition for a right annihilator, and because in rings our definition yields the two-sided annihilator, so calling it left annihilator may have been misleading.

The reader can find further information on the elementary properties of centrality in Chapter 3 of [7]. For deeper properties, the paper [9] may be consulted.

The modular commutator is a useful tool for infinite algebras, too. We shall now learn the technique of iteration, which plays a crucial role in tame congruence theory. This technique works only for finite algebras. From now on, we shall consider finite algebras in this paper, unless the context implies otherwise.

First let us clarify what is usually meant by the word *idempotent* in universal algebra. An element e of a semigroup is idempotent if $e^2 = e$, and a semigroup itself is called idempotent if every element is idempotent. This has got generalized by calling an element e of an algebra idempotent, if $f(e, \dots, e) = e$ holds for every basic operation f. The operation f is idempotent, if $f(x, \dots, x) = x$ for every x, and an algebra is idempotent, if every operation is idempotent.

On the other hand, unary functions on a set form a semigroup under composition, so such a function f is idempotent (as a semigroup element) if $f \circ f = f$, while it is idempotent (as an operation) if $f(x) = x$ for every x, that is, f is the identity map. Since this second meaning is trivial, the first usage has became accepted, and we shall adopt this terminology, too. It rarely causes confusion in practice.

Now let f be an m-ary function ($m > 1$) on a set A. Let $g_{\mathbf{b}}(x) = f(x, \mathbf{b})$. If this unary function is idempotent for every choice of $\mathbf{b}$, then f is called idempotent in its variable x. Iteration is a main tool of tame congruence theory used to construct such operations.

Suppose that the set A is finite. Then there exists a positive integer $n = n_{\mathbf{b}}$ such that $g_{\mathbf{b}}^n$ is idempotent. Setting N to be the product of the numbers $n_{\mathbf{b}}$, with $\mathbf{b}$ running over the elements of A^{m-1}, we see that $g_{\mathbf{y}}^N$ is idempotent for every $\mathbf{y}$. This m-ary function is denoted by $f_{(1)}^N$, or simply by $f_{(1)}$, and it is idempotent in its first variable. For example, if f is binary and $N = 3$, then $f_{(1)}^3(x, y) = f(f(f(x, y), y), y)$. The function $f_{(i)}$ is defined analogously. The phrase 'we iterate f in the i-th variable (to become idempotent)' will mean that we construct $f_{(i)}$. Clearly, when iterating a term or a polynomial of an algebra, we obtain a term or a polynomial, respectively.

As the first application of iteration we prove that finite quasigroups are Mal'cev (see Lemma 4.6 of [7]). Recall that a quasigroup operation on a set A is a binary function f such that $f(x, a)$ and $f(a, x)$ are permutations for every $a \in A$. We shall actually prove a slightly more general statement.

Lemma 2.10. *Let f be a ternary operation on a finite set A, and B a nonempty subset of A that is closed under f. Suppose that for every $b, c \in B$ the functions $f(x, b, c)$ and $f(b, c, x)$ are permutations of A. Then the clone generated by f contains a ternary term d satisfying $d(b, b, x) = d(x, b, b) = x$ for every $b \in B$ and $x \in A$. Moreover, if $f(x, x, x)$ is a permutation of A, then d can be chosen to be idempotent, that is, to satisfy $d(x, x, x) = x$ for all $x \in A$.*

Notice that if $\circ$ is a quasigroup operation on A, then by letting $B = A$ and $f(x, y, z) = x \circ z$ the lemma yields a Mal'cev function. So this statement is indeed a generalization of Lemma 4.6 of [7]. Its proof is also very similar. Therefore we only outline its steps, and strongly encourage the reader to fill in the details.

Proof. First, if $h(x) = f(x, x, x)$ is a permutation of A, then there is an integer n such that h^n is the identity map. Replace f by $h^{n-1} \circ f$. Then the conditions remain intact, and we have in addition that the new f is idempotent, that is, $f(x, x, x) = x$ for every $x \in A$.

Iterate f in its first variable to become idempotent, giving an operation $f^n_{(1)}$. By doing this process in the third variable we get $f^m_{(3)}$. As the only idempotent permutation is the identity map, we have

$$f(f^{n-1}_{(1)}(x, b, c), b, c) = x \quad \text{and} \quad f(b, c, f^{m-1}_{(3)}(b, c, x)) = x$$

for every $b, c \in B$ and $x \in A$. Define

$$d(x, y, z) = f(f^{n-1}_{(1)}(x, y, f^{m-1}_{(3)}(y, y, y)), y, f^{m-1}_{(3)}(y, y, z)) .$$

Then $d(x, b, b) = x$ for every $b \in B$, $x \in A$. Let any $b \in B$ be given and let $c = f^{m-1}_{(3)}(b, b, b) \in B$. By the displayed equations above, we have $f(b, b, c) = b = f(f^{n-1}_{(1)}(b, b, c), b, c))$, and as $f(x, b, c)$ is a permutation, we see that $f^{n-1}_{(1)}(b, b, c) = b$. Hence $d(b, b, x) = x$. If f is idempotent, then clearly so is d. □

Finally we introduce the concept of an *induced algebra* (see Definition 2.2 of [7]). Let $\mathbf{A}$ be an algebra and $U \neq \emptyset$ a subset of A. Consider all polynomials f of $\mathbf{A}$, which can be restricted to U, that is, which satisfy that $f(U, \ldots, U) \subseteq U$. The *induced algebra* of $\mathbf{A}$ on U (denoted by $\mathbf{A}|_U$) is defined to have underlying set U, and its basic operations are the restrictions $f|_U$ of all such polynomials f. The similar notation $\alpha|_U$ denotes the restriction of a congruence α to the set U (that is, $\alpha \cap (U \times U)$).

Let U and V be nonempty subsets of an algebra $\mathbf{A}$. We say that these subsets are *polynomially isomorphic* (see Definition 2.7 of [7]) if there exist unary polynomials f and g of $\mathbf{A}$ that are mutually inverse bijections between these subsets. The easy proof of the following claim is left to the reader (it is embedded into the pages following Definition 2.7 of [7]).

Lemma 2.11. *Let U and V be nonempty subsets of an algebra $\mathbf{A}$.*

1. *If the sets U and V are polynomially isomorphic, then the induced algebras $\mathbf{A}|_U$ and $\mathbf{A}|_V$ are isomorphic.*
2. *If A is finite, and there exists unary polynomials f and g of $\mathbf{A}$ such that $f(U) = V$ and $g(V) = U$, then U and V are polynomially isomorphic.*

3. The twin lemma

One of the starting points of tame congruence theory is the structure theorem of minimal algebras in Chapter 4 of [7]. Let $\delta < \theta$ be congruences of a finite algebra $\mathbf{C}$. We say that $\mathbf{C}$ is *minimal* with respect to the quotient $\langle \delta, \theta \rangle$ if for every $f \in \operatorname{Pol}_1(\mathbf{C})$, either f is a permutation of C, or f *collapses* θ to δ, that is, $f(\theta) \subseteq \delta$. The $\langle\delta, \theta\rangle$-*traces* of $\mathbf{C}$ are those blocks of θ that consist of more than one δ-block. The *body* B of $\mathbf{C}$ is the union of all the traces, and the *tail* T is the rest of C. Clearly, every unary polynomial that is a permutation must preserve B and T. The following Twin Lemma is the key to the whole of this paper.

Lemma 3.1. (The Twin Lemma) *Let* $\mathbf{C}$ *be minimal with respect to* $\langle\delta, \theta\rangle$*, and let* B *denote the body of* $\mathbf{C}$*. Suppose that* f *and* g *are unary* $B \times B$*-twin polynomials of* $\mathbf{C}$ *such that* f *is a permutation but* g *is not. Then* B *is a single* θ*-class, which is a union of two* δ*-classes, and* $\mathbf{C}$ *has a binary polynomial that is a semilattice operation on* $B/\delta|_B$*.*

Proof. We shall call $B \times B$-twin polynomials of $\mathbf{C}$ simply *body-twins* throughout the paper. To show the reader what is going on, we shall use a special language (and enclose the formal arguments in brackets). By the minimality of $\mathbf{C}$, any unary polynomial can be of two different *characters:* it is either *collapsing* (collapses all traces to δ), or *permutational.* Both characters are clearly preserved when going down to $\mathbf{C}/\delta$. Therefore we may assume in the proof that $\delta = 0_C$. Thus, the word collapsing now means that the polynomial is constant on every trace.

Let $f(x) = h'(x, \mathbf{c})$ and $g(x) = h'(x, \mathbf{d})$. Since we can move from $\mathbf{c}$ to $\mathbf{d}$ coordinate by coordinate, we may assume that h' is binary. Iterate h' in its first variable to get a polynomial $h = h'_{(1)}$ satisfying that $h(x, a)$ is an idempotent function for every $a \in C$. From the process of iteration we see that $h(x, c)$ is a permutation, but $h(x, d)$ is not. We also know that $c, d \in B$.

Draw the multiplication table of h, where the columns correspond to the mappings $h(x, a)$, and the rows correspond to the mappings $h(a, y)$. We call two columns (or rows) θ-related if they are of the form $h(x, u)$ and $h(x, v)$ (or $h(u, y)$ and $h(v, y)$), where $(u, v) \in \theta$. All columns correspond to idempotent maps, and as the only idempotent permutation is the identity map, every column is the identity, or it is collapsing. In particular, the column of c is the identity map, and the column of d is collapsing.

First we show that there is only one permutational column. Indeed, suppose that there exists an element $b \neq c$ such that the column of b [the map $h(x, b)$] is also a permutation, hence the identity map. Then we have two equal columns, hence every row contains two equal elements [as $h(x, b) = x = h(x, c)$ for every x], and therefore every row is collapsing. In other words, any two θ-related columns are equal [$u \;\theta\; v$ implies $h(x, u) = h(x, v)$]. In particular, c and d are in different traces. Since they are in the body, there exist $c' \;\theta\; c$ and $d' \;\theta\; d$ such that $c' \neq c$ and $d' \neq d$. Look at the diagonal [the unary polynomial $h(x, x)$] of the table. It collapses the trace of d [since $h(d, d) = h(d', d) = h(d', d')$], and is the identity map on the trace of c [since $h(c, c) = c \neq c' = h(c', c) = h(c', c')$]. This contradiction shows that the only permutational column is the column of c.

Next we show that B has only two elements. To get a contradiction suppose that there exist elements u and v such that $u\ \theta\ v$ and c, u, v are pairwise different. The columns of u and v are collapsing, hence any two θ-related rows must have the same character [since if $u'\ \theta\ v'$ and $h(u',u)=h(u',v)$, then $h(v',u)=h(u',u,)=h(u',v)=h(v',v)$]. Let N be the trace containing c. As $h(c,c)=c$, we have $h(N,N)\subseteq N$, let us consider the subtable of h on N. All the columns except the column of c are constant, so the rows are all equal, except for the entry below c, where all elements of N are listed. Therefore at most one of the rows can be a permutation, so there is one which is collapsing. As the characters of the rows are the same, every row of the subtable is constant, so the whole table is constant, which is impossible. [Formally: let $c'\neq c$, $c'\in N$, and $c''=h(c,c')\in N$. Then $h(c'',c)=c''=h(c,c')=h(c'',c')$, hence $h(c'',y)$ is collapsing, hence $h(c',y)$ and $h(c,y)$ are collapsing, hence $c=h(c,c)=h(c,c')=h(c',c')=h(c',c)=c'$, a contradiction.]

Thus $\mathbf{C}$ indeed has only one, two-element trace, which is $N=\{c,d\}$. We have either $h(c,d)=h(d,d)=c$, or $h(c,d)=h(d,d)=d$. In the second case h is a semilattice operation on N. In the first case $h(d,y)$ switches c and d, hence $h(x,h(d,y))$ is a semilattice operation. □

In the rest of this section we sketch those applications of the Twin Lemma, which help describe the structure of minimal algebras. First assume that $\mathbf{C}$ is a $\langle\delta,\theta\rangle$-minimal algebra satisfying the conclusion of the Twin Lemma. Then the induced algebra on $B/\delta|_B$ has a semilattice operation, and is therefore nonabelian. By Lemma 4.8 of [7], this two-element algebra is polynomially equivalent to a Boolean algebra, or to a lattice, or to a semilattice. Accordingly we say that the type of $\mathbf{C}$ with respect to $\langle\delta,\theta\rangle$ is **3**, **4**, or **5** (these are the nonabelian types). The structure of these algebras is described in Lemmas 4.15 and 4.17 of [7]. We suggest that the reader read these statements and their proofs now. The essence of the result is this.

Lemma 3.2. *Let* $\mathbf{C}$ *be a* $\langle\delta,\theta\rangle$*-minimal algebra of nonabelian type. Then the body* B *is a single trace (*θ*-class), which is the union of two* δ*-classes* I *and* O*, where* $I=\{1\}$ *is a one-element set. Furthermore, there exists a binary polynomial* p *of* $\mathbf{C}$ *satisfying the following conditions.*

(1) *For all* $x\in C-\{1\}$, $\langle\{1,x\},p\rangle$ *is a semilattice with neutral element* 1*, i.e.,* $p(x,1)=p(1,x)=p(x,x)=x$ *for all* $x\in C$.
(2) *For all* $x\in C$ *such that* $x\neq 1$ *and for all* $u\in O$, $p(x,u)\ \delta\ p(u,x)\ \delta\ x$.
(3) *For all* x,y *in* C, $p(x,p(x,y))=p(x,y)$.

The reader may try to prove this using iteration. If p and 1 satisfy (1) – (3) above, then p is called a *pseudo-meet operation,* and the element 1 a *neutral element* with respect to p. In the type **3** and **4** cases the body is of the form $\{0,1\}$, where both elements are neutral (1 with respect to a pseudo-meet, 0 with respect to a pseudo-join operation). Conversely, suppose that $1\neq 1'$ are two neutral elements with respect to p and p', respectively. Then both are in a one-element δ-block contained in B. Thus $B = \{1, 1'\}$, and the polynomials p and p' yield a lattice structure on B, so the type is **3** or **4**. Therefore in the type **5** case there is a unique neutral element, and this element works for every pseudo-meet operation (with respect to $\langle\delta, \theta\rangle$). We shall use this observation in Section 5.

Now let us investigate the case when body-twin polynomials of $\mathbf{C}$ always have the same character. In this case, as we shall see later, θ/δ is an abelian congruence, so we shall say that $\mathbf{C}$ is of *abelian type* with respect to $\langle\delta, \theta\rangle$. We prove a weaker statement first, the main idea of which is from Lemma 4.27 of [7].

Definition 3.3. Let $\mathbf{A}$ be any algebra. Define the *twin congruence* $\tau(\mathbf{A})$ by the following rule: $(a, b) \in \tau(\mathbf{A})$ if and only if for every binary polynomial f of $\mathbf{A}$ we have that $f(a, x)$ is a permutation if and only if $f(b, x)$ is a permutation.

It is straightforward to see that this is indeed a congruence.

Theorem 3.4. *Let $\mathbf{C}$ be minimal with respect to $\langle\delta, \theta\rangle$ of abelian type, and let B be the body of $\mathbf{C}$. Then the following hold.*

(1) *$\tau(\mathbf{C})$ is the largest binary relation β satisfying $C^2(\beta, \theta; \delta)$.*
(2) *B is contained in a single $\tau(\mathbf{C})$-class.*
(3) *$\theta \leq \tau(\mathbf{C})$.*
(4) *$(\delta : \theta) = \tau(\mathbf{C})$, in particular, we have $C(\theta, \theta; \delta)$.*

Proof. We shall prove (4) later in this section. First we show (1). If a pair (a, b) is not in $\tau(\mathbf{C})$, then there exists a binary polynomial f such that $f(a, x)$ is a permutation, but $f(b, x)$ is not. By the minimality of $\mathbf{C}$, $f(b, x)$ is collapsing, so for some $(c, d) \in \theta - \delta$ we get that $f(b, c)\ \delta\ f(b, d)$. On the other hand, $f(a, c)\ \theta - \delta\ f(a, d)$, and this is a failure of $C^2(\{(a, b)\}, \theta; \delta)$. Conversely, if (a, b) is in $\tau(\mathbf{C})$, then we have to show for every $f \in \operatorname{Pol}_2(\mathbf{C})$ and $c\ \theta\ d$ that $f(a, c)\ \delta\ f(a, d)$ implies $f(b, c)\ \delta\ f(b, d)$. Suppose instead that $f(b, c)\ \theta - \delta\ f(b, d)$. Then $f(b, y)$ is not collapsing, so it is a permutation by the minimality of $\mathbf{C}$. On the other hand, $c\ \theta - \delta\ d$, so $f(a, y)$ is not a permutation, which is a contradiction. Thus (1) is proved.

Statement (2) is an obvious consequence of the Twin Lemma. To prove (3) notice that $C^2(\delta, \theta; \delta)$ obviously holds, so $\delta \leq \tau(\mathbf{C})$ by (1). But $\theta \subseteq B^2 \cup \delta$, so we are done by (2). ☐

Call $\mathbf{C}$ of type **2** with respect to $\langle\delta, \theta\rangle$ if the induced algebra on the body B of $\mathbf{C}$ has a Mal'cev polynomial, and call $\mathbf{C}$ of type **1** otherwise.

First we look at the type **1** case. Let f be a k-ary polynomial, and $N_1, \ldots, N_k$ $\langle\delta, \theta\rangle$-traces of $\mathbf{C}$. We show that f depends on at most one variable on $N_1 \times \cdots \times N_k$ modulo δ. Indeed, we may assume by factoring δ out that $\delta = 0$. Suppose, to get a contradiction, that f depends on at least two variables, say on the first and second one, on $N_1 \times \cdots \times N_k$. Then we can fix all variables other than the first one in $N_1 \times \cdots \times N_k$ so that the resulting unary polynomial is not constant on N_1, hence it is a permutation of C. By the Twin Lemma, if we fix these variables arbitrarily in B, we still get a permutation. The same holds for the second variable of f. Therefore fixing all but the first two variables arbitrarily in B we get a binary polynomial that is a quasigroup operation on B. By Lemma 2.10 we have a Mal'cev polynomial on B. This contradiction proves our statement.

In particular, the induced algebras on the traces are essentially unary modulo δ. We now prove the type **1** case of statement (4) of Theorem 3.4. It is sufficient to prove $C(\tau(\mathbf{C}), \theta; \delta)$. Suppose that $(a, b) \in \tau(\mathbf{C})$, $\mathbf{c}\ \theta\ \mathbf{d}$, and $f(a, \mathbf{c})\ \theta - \delta\ f(a, \mathbf{d})$ for some $k+1$-ary polynomial f. We want to show that $f(b, \mathbf{c})\ \theta - \delta\ f(b, \mathbf{d})$. Since we can move c_i and d_i within their δ-classes without changing anything, we may assume, by possibly fixing some variables of f, that $c_i, d_i \in N_i$ for some trace N_i, for all i. Now $f(a, \mathbf{x})$ depends on at most one variable on the product of these traces modulo δ, say on the first one. Therefore we have $f(a, c_1, c_2, \ldots, c_k)\ \theta - \delta\ f(a, d_1, c_2, \ldots, c_k)$. By $C^2(\tau(\mathbf{C}), \theta; \delta)$ we get that $f(b, c_1, c_2, \ldots, c_k)\ \theta - \delta\ f(b, d_1, c_2, \ldots, c_k)$. Hence $f(b, \mathbf{x})$ depends on its first variable on $N_1 \times \cdots \times N_k$. Therefore it does not depend on the others, implying $f(b, \mathbf{c})\ \theta - \delta\ f(b, \mathbf{d})$ as desired.

We do not prove more statements on the structure of minimal algebras of type **1** (there are no more statements in [7] either), but we should keep in mind that the Twin Lemma and Theorem 3.4 give extra information on these algebras. We shall also investigate a special case in Section 4, namely E-minimal algebras of type **1**.

The structure of minimal algebras of type **2** is given by Lemma 4.20 of [7]. We now quote this statement, and prove it using our machinery.

Lemma 3.5. *Let* $\mathbf{C}$ *be a* $\langle \delta, \theta \rangle$*-minimal algebra of type* **2**, *and let* B *denote the body of* $\mathbf{C}$. *Then every trace is polynomially equivalent to a vector space modulo* δ, *and* $\mathbf{C}$ *has a ternary polynomial* d *satisfying:*

(1) $d(x, x, x) = x$ *for all* $x \in C$.
(2) $d(b, b, y) = y = d(y, b, b)$ *for all* $b \in B$ *and* $y \in C$.
(3) *for every* $a, b \in B$, *the unary polynomials* $d(x, a, b)$, $d(a, x, b)$, $d(a, b, x)$ *are permutations of* C.
(4) B *is closed under* d.
(5) *any two* $\langle \delta, \theta \rangle$*-traces* N *and* N' *are polynomially isomorphic.*

Every ternary polynomial of $\mathbf{C}$ *satisfying* (1) *and* (2) *also satisfies* (3) *and* (4).

Proof. Since the type is **2**, there exists a ternary polynomial f of $\mathbf{C}$ that is a Mal'cev function on B. Thus $f(b, b, x) = x$ for all $b, x \in B$. This means that $f(b, b, x)$ is a permutation on B, and as $\mathbf{C}$ is minimal, it is a permutation on $\mathbf{C}$. By the Twin Lemma, $f(b, c, x)$, and similarly, $f(x, b, c)$ are permutations for every $b, c \in B$. Finally, $f(x, x, x)$ is also a permutation, as it is the identity map on B. Therefore Lemma 2.10 yields a ternary polynomial d satisfying (1) and (2).

Now let N be a $\langle\delta,\theta\rangle$-trace. Then Theorem 3.4 implies $C^2(\theta,\theta;\delta)$, so the induced algebra on N/δ is abelian by Lemma 2.8. As N is a congruence-class and d is idempotent, we have $d(N,N,N)\subseteq N$. Therefore the induced algebra on N/δ is Mal'cev, hence it is affine by Lemma 2.6. It is easy to compute, using the minimality of $\mathbf{C}$, that the corresponding module must actually be a vector space.

Now suppose that a ternary polynomial d satisfies (1) and (2). Let $a,b\in B$. The mappings $d(x,a,a)=x$, and $d(a,a,x)=x$ are permutations on C hence so are $d(x,a,b)$ and $d(a,b,x)$ by the Twin Lemma. Let $b'\ \theta-\delta\ b$. Then $d(b',b,b)=b'\ \theta-\delta\ b=d(b',b',b)$, so $d(b',x,b)$ is a permutation. Therefore $d(a,x,b)$ is also a permutation by the Twin Lemma, proving (3). As every permutation preserves the body, (4) follows from (3).

Finally, let N and N' be traces, $a\in N$, $b\in N'$. Then the permutation $d(a,b,x)$ maps b to a, so it maps $N'=b/\theta$ into $N=a/\theta$. As C is finite, any two traces have the same size, and all these maps are bijections between the traces. Hence Lemma 2.11 finishes the proof. □

A ternary operation satisfying $(1)-(4)$ is called a *pseudo-Mal'cev operation.* Notice that, in particular, we have obtained Pálfy's Theorem 4.7 in [7], too, about $\langle 0,1\rangle$-minimal algebras, as well as the type **2** case of statement (4) of Theorem 3.4, which is now a direct consequence of Lemma 2.9.

The next statement (found basically in Chapter 4 of [7]) will be used later.

Lemma 3.6. *Let* $\mathbf{C}$ *be a type* **2** *minimal algebra with respect to a quotient* $\langle\delta,\theta\rangle$*, let* B *denote the body and* T *the tail of* $\mathbf{C}$*, and let* d *be a pseudo-Mal'cev operation of* $\mathbf{C}$*. Then the following hold.*

1. *The body is a class of the twin congruence.*
2. *Let* $b\in B$ *and* $t\in T$*. Then* $d(t,t,b),d(b,t,t),d(t,b,t)\in T$*.*

Proof. Let $b\in B$ and $t\in T$. We have $d(x,b,b)=x$, so this is a permutation. On the other hand, $d(b,b,t)=t$, so $d(x,b,t)$ maps b to t, and therefore it cannot be a permutation. Thus the binary polynomial $d(x,b,y)$ shows that (b,t) is not in $\tau(\mathbf{C})$, proving (1).

The first two inclusions in (2) are basically proved in the book [7] (see Lemma 4.25 and the proof of Lemma 4.27 (4ii)). The proof of the third inclusion is very similar. We present this argument, and the reader will surely be able to deduce the proofs of the first two inclusions.

Suppose that $a=d(t,b,t)\in B$. Consider the polynomial

$$h(x)=d(d(d(x,b,t),b,t),a,x)$$

of $\mathbf{C}$. Then $h(t)=d(d(a,b,t),b,t)\ \tau(\mathbf{C})\ d(d(b,b,t),b,t)=a$. Thus $h(t)\in B$ by (1), and therefore h cannot be a permutation, since it maps an element of the tail into the body. On the other hand, if $x\ \theta\ b$, then $d(x,b,t)\ \theta\ d(b,b,t)=t$, but this element is in the tail, and therefore we have $d(x,b,t)\ \delta\ t$. Therefore $h(x)\ \delta\ d(a,a,x)=x$. This shows that h is not collapsing either, contradicting the minimality of $\mathbf{C}$. Therefore $d(t, b, t) \in T$. To prove $d(t, t, b) \in T$ one has to use a similar argument using a modified form of h that is found in the proof of Lemma 4.25 of [7]. □

Statement (1) is false in the type **1** case (take a 3-element set with no operations). As an exercise, we suggest the reader to prove, based on this last lemma, statement (4) of Lemma 4.27 in [7].

4. E-minimal algebras

A finite algebra $\mathbf{C}$ is called E-minimal, if it is minimal with respect to all of its prime congruence quotients. These algebras play an important role in the description of minimal algebras of type **2**, as shown in the second part of Chapter 4 of [7]. Namely, it is proved there that the induced algebra on the body of such a minimal algebra is always E-minimal.

Lemma 4.29 of [7] states that if an E-minimal algebra has a nonabelian quotient, then it is a two-element algebra. Clearly, all two-element algebras are E-minimal, so there is nothing more one can say about the nonabelian case. E-minimal algebras of type **2** have been completely described in Theorem 13.9 of [7]. In this section we shall provide a similar characterization of the type **1** case.

As shown in Lemma 4.28 of [7], $\mathbf{C}$ is E-minimal iff every idempotent unary polynomial of $\mathbf{C}$ is either constant or the identity map. This equivalence is a straightforward consequence of the fact that every minimal set is the range of an idempotent unary polynomial (see Theorem 2.11 of [7]).

Lemma 4.1. *Let $\mathbf{C}$ be an E-minimal algebra. Suppose that f and g are unary $C \times C$-twins of $\mathbf{C}$ such that f is a permutation but g is not. Then $\mathbf{C}$ is a two-element algebra, and it has a binary polynomial that is a semilattice operation.*

Proof. Let δ be a maximal congruence of $\mathbf{C}$. Then the $\langle \delta, 1_C \rangle$-body of $\mathbf{C}$ is C, and therefore f and g are body-twins. So by the Twin-Lemma, the quotient $\langle \delta, 1_C \rangle$ is nonabelian, and we are done by Lemma 4.29 of [7] stated above. □

Corollary 4.2. *Every prime quotient of an E-minimal algebra has the same type.*

Proof. Let $\mathbf{C}$ be an E-minimal algebra. If a nonabelian quotient occurs in $\mathbf{C}$, then by Lemma 4.29 of [7], $\mathbf{C}$ is a two-element algebra, and we are done. If $\mathbf{C}$ has a prime quotient of type **2**, then Lemma 4.1 shows that the twin congruence must be 1_C. By Lemma 3.6, the body with respect to this quotient is C, so C has a Mal'cev polynomial. Therefore there is no type **1** quotient. □

Note that Corollary 4.2 is generalized in Section 5 (see Theorem 5.2). This common type of all the quotients is called the type of the E-minimal algebra itself.

By Lemma 4.1 the twin congruence is 1_C for an E-minimal algebra $\mathbf{C}$ of abelian type. Hence Theorem 3.4 (4) yields that such algebras are left nilpotent. This is the statement of Theorem 4.36 of [7] for the type **2** case, and of Corollary 4.11 of [9] for the type **1** case. We call attention to Lemma 4.10 of [9], which is the solvable case of Lemma 4.1 above, with a different proof. Another important exercise, also mentioned in [9], is to deduce from Lemma 4.1 that every finite algebra in the variety generated by a finite E-minimal algebra of abelian type is also E-minimal (and has the same type as the generator).

So far, we have been trying to provide 'easy' arguments for the statements we stated. The reader is encouraged to modify the proof of the Twin Lemma to prove Lemma 4.1 directly (that is, without referring to the structure of minimal algebras of nonabelian type). [Here is the idea. The only part in the proof of the Twin Lemma using that f and g are body-twins is the fourth paragraph showing that there is only one permutational column. Let K denote those elements of C that correspond to collapsing columns. By E-minimality, these columns are constants. If there are two permutational columns, then, as in the other proof, all rows are collapsing, and the diagonal is a permutation. Hence the diagonal permutes K, and the reader will be able to get a quick contradiction by showing that an idempotent power of the row of c must have a fixed point inside as well as outside K.]

We included this sketch partly because we wonder if it leads to a common generalization of the Twin Lemma, and of Lemma 4.1. From Theorem 3.4 we know that in a $\langle \delta, \theta \rangle$-minimal algebra we have that all twins of permutations are permutations if and only if $C(1_C, \theta; \delta)$ holds. Thus, we are looking for conditions forcing $C(1_C, \theta; \delta)$. Left nilpotence is obviously sufficient. On the other hand, if $\mathbf{C}$ is **solvable** of type **2**, then $\mathbf{C}$ is full-bodied by Lemma 4.27 (4) of [7], and therefore we still have $C(1_C, \theta; \delta)$ by Theorem 3.4. Unfortunately, the following algebra shows that the same conclusion does not hold in the type **1** case. Let $\mathbf{A} = \langle \{0, 1, 2\}, \circ \rangle$, where the binary operation $\circ$ is given by

$\circ$	0	1	2
0	0	0	0
1	1	1	0
2	2	2	2

It is clear from the table that twins of permutations are not permutations. We leave as an exercise the verification of the rest of the details.

Now we present our characterization of the clones of E-minimal algebras of type **1**. Let C be any finite set of at least two elements, and let $0_C = \delta_0 < \delta_1 < \cdots < \delta_k = 1_C$ be a chain of equivalence relations of C with $k \geq 1$. Our purpose is to define a clone $E(\delta_0, \ldots, \delta_k)$ such that the algebra $\mathbf{E}(\delta_0, \ldots, \delta_k) = \langle C, E(\delta_0, \ldots, \delta_k) \rangle$ is an E-minimal algebra of type **1**. Our result will then state that an algebra is E-minimal of type **1** if and only if it is a reduct of an algebra of the form $\mathbf{E}(\delta_0, \ldots, \delta_k)$ for an appropriate chain of equivalence relations. At the end of this section we shall try to explain and illustrate the usefulness of this result.

To define the clone $E(\delta_0, \dots, \delta_k)$, call a unary function f on the set C *collapsing,* if $f(\delta_s) \subseteq \delta_{s-1}$ for every $1 \leq s \leq k$. If f is n-ary, then we say that f *is collapsing in its i-th variable,* if no matter how we fix all other variables in C, the resulting unary function is collapsing. Similarly, we say that f *is permutational in its i-th variable,* if no matter how we fix all other variables in C, the resulting unary function is a permutation that preserves every equivalence relation δ_s. (Obviously, every permutational f satisfies that $f(\delta_s) \not\subseteq f(\delta_{s-1})$ for every $1 \leq s \leq k$.) Now let $E(\delta_0, \dots, \delta_k)$ be the set of all functions of C that are either collapsing in every variable, or permutational in one variable, and collapsing in all the others.

Lemma 4.3. *The set $E(\delta_0, \dots, \delta_k)$ is a clone, that is, it is closed under composition, and contains the projections.*

Proof. Clearly, an i-th projection is permutational in its i-th variable, and is collapsing in the others. Next let $f \in E = E(\delta_0, \dots, \delta_k)$ be n-ary, and $g_1, \dots, g_n \in E$ be unary. We show that the unary composition $h = f(g_1, \dots, g_n)$ is a permutation if f is permutational in its i-th variable and g_i is a permutation, and h is collapsing in all other cases. Indeed, let $(a, b) \in \delta_s - \delta_{s-1}$ for some $1 \leq s \leq k$, and consider

$$\begin{aligned} a_0 &= f(g_1(a), g_2(a), \dots, g_n(a)), \\ a_1 &= f(g_1(b), g_2(a), \dots, g_n(a)), \\ &\cdots \\ a_n &= f(g_1(b), g_2(b), \dots, g_n(b)). \end{aligned}$$

If f is collapsing in its j-th variable, or if g_j is collapsing, then we clearly have $a_{j-1} \ \delta_{s-1} \ a_j$. Therefore h is collapsing unless there exists an i such that g_i is a permutation, and f is permutational in the i-th variable. Assume this. Then by $f \in E$, f is collapsing in all other variables, so we have $a_{j-1} \ \delta_{s-1} \ a_j$ for every j except for $j = i$. But $f(g_1(b), \dots, g_{i-1}(b), g_i(x), g_{i+1}(a), \dots, g_n(a))$ is a permutation, and therefore $a_{i-1} \ \delta_s - \delta_{s-1} \ a_i$, thus by transitivity, $h(a) \ \delta_s - \delta_{s-1} \ h(b)$. Since any pair of equal elements is obviously in δ_{s-1}, we see that $h(a) \neq h(b)$. But every pair (a, b) with $a \neq b$ is contained in some $\delta_s - \delta_{s-1}$, so we have proved that h is indeed a permutation.

Now let $f \in E$ be n-ary, and $g_i \in E$ be m-ary, for $1 \leq i \leq n$. We have to prove that the m-ary composition $h = f(g_1, \dots, g_n) \in E$. Let $1 \leq j \leq m$, and fix all other variables of h arbitrarily. Then we arrive at the situation investigated in the previous paragraph. Thus we get a collapsing function unless there is an i such that f is permutational in its i-th variable, and g_i is permutational in its j-th variable, in which case h is permutational in its j-th variable. This cannot happen for two different values of j, since $f, g_i \in E$ for every i. Thus $h \in E$ as stated. □

Now let $\mathbf{E}(\delta_0, \dots, \delta_k) = \langle C, E(\delta_0, \dots, \delta_k) \rangle$.

Theorem 4.4. *The algebra* $\mathbf{E}(\delta_0, \ldots, \delta_k)$ *is E-minimal of type* $\mathbf{1}$ *. Conversely, if* $\mathbf{C}$ *is a finite E-minimal algebra of type* $\mathbf{1}$ *, then it is a reduct of* $\mathbf{E}(\delta_0, \ldots, \delta_k)$ *for any choice of congruences* $0_C = \delta_0 \prec \cdots \prec \delta_k = 1_C$ *of* $\mathbf{C}$*.*

Proof. First notice that all the polynomials of $\mathbf{E} = \mathbf{E}(\delta_0, \ldots, \delta_k)$ are terms, since $E(\delta_0, \ldots, \delta_k)$ contains the constant functions. Now let e be an idempotent unary polynomial of $\mathbf{E}$ that is not constant. Let $a \neq b$ be elements in the range of e, and choose s to satisfy $(a, b) \in \delta_s - \delta_{s-1}$. Then e does not collapse δ_s to δ_{s-1}, and therefore it is a permutation. Thus $\mathbf{E}$ is indeed E-minimal. It is straightforward to see that for every s, the induced algebra on any δ_s-block is essentially unary modulo δ_{s-1}, so the type is indeed $\mathbf{1}$.

Now let $\mathbf{C}$ be an E-minimal algebra of type $\mathbf{1}$, and pick any maximal chain of congruences $0_C = \delta_0 \prec \cdots \prec \delta_k = 1_C$ of $\mathbf{C}$. We have to prove that every operation f of $\mathbf{C}$ belongs to $E(\delta_0, \ldots, \delta_k)$. Indeed, f is either collapsing or a permutation in every variable. By Lemma 4.1, this character does not depend on the way the other variables are fixed. Finally, because the type is $\mathbf{1}$, f cannot be permutational in two variables. Thus f indeed belongs to $E(\delta_0, \ldots, \delta_k)$. □

Let us explain why such a result as Theorem 4.4 can be useful. Notice that it is not easy to construct minimal algebras in general. When the desired operation tables are produced, it is always a nontrivial question if some complicated composition of the basic operations spoils minimality. So one has to compute all unary polynomials, and that is not always possible, not even with a computer.

On the other hand, to construct E-minimal algebras, one can simply pick any chain of equivalence relations, and choose the basic operations from the set $E(\delta_0, \ldots, \delta_k)$. Thus in the type $\mathbf{1}$ case, E-minimal algebras provide a useful way of testing conjectures for the general nilpotent or solvable case. For example, Examples 1 and 2 of [9], showing that left nilpotence does not imply right nilpotence in finite algebras, are E-minimal. Example 1 is also interesting, because it generates a residually large variety (see [11]). We present it here as an illustration on how to use Theorem 4.4 to construct an E-minimal algebra.

Let $\mathbf{A}$ denote the algebra $\langle \{0, 1, 2, 3\}, \circ, f \rangle$, where the binary operation $\circ$ and the unary operation f are given by the following table:

$\circ$	0	1	2	3	f
0	0	0	0	0	1
1	1	1	1	2	0
2	2	2	2	1	2
3	3	3	3	3	3

To show that this algebra is E-minimal of type $\mathbf{1}$, let $\delta_0 = 0_A$, $\delta_2 = 1_A$, and denote by δ_1 the partition with blocks $\{0, 1, 2\}$, and $\{3\}$. Then f is a permutation preserving δ_1. The operation $\circ$ is permutational in its first variable: indeed, all columns are permutations preserving δ_1. It is collapsing in its second variable, since all rows have range in a δ_1-block and all rows are constant on the blocks of δ_1. So we could construct a minimal (moreover, E-minimal) algebra without having to compute all unary polynomials.

5. Multiminimal algebras

In this section we investigate the situation when an algebra is minimal with respect to more than one quotient. This condition is weaker than E-minimality, but we are still able to prove strong consequences. Our main result is Theorem 5.2. Before formulating it we investigate the nonabelian case (the results presented in the next theorem have some overlap with the results in [1]). Recall that a neutral element of a minimal algebra of nonabelian type is a neutral element 1 with respect to a pseudo-meet operation, and that the neutral element is unique in the type **5** case (see Lemma 3.2 and the remarks following it).

Theorem 5.1. *Let* $\mathbf{C}$ *be a* $\langle \delta, \theta \rangle$*-minimal algebra of nonabelian type. Let* $\tau = \tau(\mathbf{C})$ *be the twin congruence. Then the following hold.*

(1) *If* 1 *is a neutral element of* $\mathbf{C}$*, then the twin congruence is the largest congruence of* $\mathbf{C}$ *that has* $\{1\}$ *as a congruence-class.*
(2) *The twin congruence has a unique cover* τ^* *(so it is meet-irreducible).*
(3) $\langle \tau, \tau^* \rangle$ *and* $\langle \delta, \theta \rangle$ *are perspective prime quotients* ($\theta \vee \tau = \tau^*$*, and* $\theta \wedge \tau = \delta$*).*
(4) *The algebra* $\mathbf{C}$ *is minimal with respect to* $\langle \tau, \tau^* \rangle$*.*
(5) *The* $\langle \tau, \tau^* \rangle$*-type is the same as the* $\langle \delta, \theta \rangle$*-type.*
(6) *The* $\langle \tau, \tau^* \rangle$*-body contains the* $\langle \delta, \theta \rangle$*-body. If the type is* **3** *or* **4***, then the two bodies are equal. If the type is* **5***, then the two neutral elements are equal.*

Proof. Let p be a pseudo-meet operation with neutral element 1. Define β by

$$c\,\beta\,d \iff (\forall f \in \mathrm{Pol}_1(\mathbf{C}))(f(c) = 1 \iff f(d) = 1)\,.$$

This is clearly the largest congruence of $\mathbf{C}$ which has $\{1\}$ as a congruence-class. First we show that $\tau \leq \beta$. Let f be a unary polynomial such that $f(c) = 1$, $f(d) \neq 1$ for some (c, d). Then $p(f(c), y) = y$ is a permutation, but $p(f(d), y)$ is not, since it collapses 1 and $f(d)$. Therefore (c, d) is not in τ. To show $\beta \leq \tau$, suppose that there is a pair $(c, d) \in \beta - \tau$. Then there exists a binary polynomial g such that $g(c, y)$ is a permutation, and $g(d, y)$ is not. Thus, $g(c, b) = 1$ for some b. Note that b must be in the body B. Now $g(x, b)$ maps c to 1, so by $(c, d) \in \beta$ we have that $g(d, b) = 1$. As $g(d, y)$ is collapsing, this means that $g(d, b/\theta) \subseteq 1/\delta = \{1\}$, that is, $g(d, b') = 1$ for every $b' \in B$. By $(c, d) \in \beta$ again we get that $g(c, b') = 1$ for every $b' \in B$. This is a contradiction, since $g(c, y)$ is a permutation, and B has at least two elements. Thus (1) is proved.

Next we show that $\tau^* = \tau \vee \theta$ is the unique cover of τ. Indeed, suppose that $\alpha > \tau$ is any congruence. Then, by (1), α contains a pair $(1, a)$ with $a \neq 1$. Let $1 \neq u \in B$.

Then $u = p(u, 1)\ \alpha\ p(u, a)\ \delta\ a$ by Lemma 3.2. As δ is a congruence of which $\{1\}$ is a block, we have $\delta \leq \tau$ by (1). Therefore, $1\ \alpha\ u$. But the congruence θ is generated by $(1, u)$ and δ, so we have shown that $\alpha \geq \tau \vee \theta = \tau^*$, proving (2).

It is clear from Lemma 3.2 that θ covers δ. We have already proved $\tau^* = \tau \vee \theta$ and $\delta \leq \tau$, so (3) follows from the fact that θ is not below τ. If f is a unary polynomial that is not a permutation, then $f(\theta) \subseteq \delta$, so $f(\tau^*) \subseteq f(\tau) \vee f(\theta) \subseteq \tau$, showing (4).

We have $B = 1/\theta \subseteq 1/\tau^*$. Hence $1/\tau^* \neq 1/\tau = \{1\}$, and therefore $N = 1/\tau^*$ is a $\langle \tau, \tau^* \rangle$-trace. So B is contained in the $\langle \tau, \tau^* \rangle$-body. Thus p demonstrates that $\langle \tau, \tau^* \rangle$ is of nonabelian type, hence the trace N is the $\langle \tau, \tau^* \rangle$-body, containing B. So if the $\langle \tau, \tau^* \rangle$-type is not **5**, then both bodies are two-element sets, and therefore they are equal. In the type **5** case the element 1 is a neutral element for $\langle \tau, \tau^* \rangle$ (since p induces a meet operation on $N/(\tau|_N)$). As the $\langle \tau, \tau^* \rangle$-neutral element is uniquely determined, all the statements are proved. □

Theorem 5.2. *Let* $\mathbf{C}$ *be a finite algebra that is minimal with respect to more than one quotient. Then the following hold.*

(1) *The type of* $\mathbf{C}$ *with respect to any of these quotients is the same.*
(2) *If this type is* **2**, **3**, *or* **4**, *then the corresponding bodies are also the same.*
(3) *If the type is* **5**, *then the neutral element* 1 *is the same for all quotients, and there is a largest one among the bodies with respect to the quotients for which* $\mathbf{C}$ *is minimal.*

Proof. Let $\mathbf{C}$ be a minimal algebra with respect to a quotient $\langle \delta, \theta \rangle$, let B denote the body and T the tail of $\mathbf{C}$. Assume that $\mathbf{C}$ is also minimal with respect to another quotient $\langle \delta', \theta' \rangle$ with body B' and tail T'. First consider the case, where the first quotient is nonabelian. By Lemma 3.2, the algebra $\mathbf{C}$ has a pseudo-meet operation p with neutral element $1 \in B$. Let $a \in C - \{1\}$. Now $p(a, a) = a = p(a, 1)$, so $p(a, y)$ is not a permutation. Similarly, $p(x, a)$ is not a permutation either. Hence these unary polynomials collapse θ' to δ'. Thus, with $c\ \theta'\ d$, $c, d \neq 1$, we have that $c = p(c, c)\ \delta'\ p(c, d)\ \delta'\ p(d, d) = d$. So B' must be the union of exactly two blocks of δ', one of which is $\{1\}$. Moreover, p gives a meet operation on $B'/\delta'|_{B'}$. Therefore $\mathbf{C}$ is nonabelian with respect to $\langle \delta', \theta' \rangle$ also. Thus Theorem 5.1 applies to both quotients. Since the twin congruence and its unique cover depend only on the algebra $\mathbf{C}$, and not on the quotient $\langle \delta, \theta \rangle$ or $\langle \delta', \theta' \rangle$, we have all statements stated in the theorem.

Now assume that the type with respect to $\langle \delta, \theta \rangle$ is **2**, let d be a pseudo-Mal'cev operation and N' a θ'-trace. We prove that $N' \subseteq B$. To get a contradiction suppose that there exists an element $t \in N' - B$. Let $t\ \theta' - \delta'\ t'$, so $t' \in N'$. By Lemma 3.6 (2) we see that the unary polynomials $d(x, t, t)$, $d(t, x, t)$, $d(t, t, x)$ map B to T, and therefore they are not permutations. Applying the Twin Lemma for B' we see that $d(t', x, t)$ and $d(t', t', x)$ are not permutations either. Therefore these maps collapse θ' to δ', hence

$$t = d(t, t, t)\ \delta'\ d(t', t, t)\ \delta'\ d(t', t', t)\ \delta'\ d(t', t', t') = t' ,$$

which is a contradiction. Thus $N' \subseteq B$. To show that the $\langle\delta', \theta'\rangle$-type is **2** choose two elements c and d of N' that are not δ'-related. Then $d(c, d, d) = c$ and $d(d, d, d) = d$ holds by $N' \subseteq B$ and the properties of the pseudo-Mal'cev operation. Hence d depends on its first variable on N' modulo δ'. From $d(c, c, d) = d$ and $d(c, c, c) = c$ we see that d also depends on its third variable, hence the induced algebra on N' is not essentially unary, and so the the type with respect to $\langle\delta', \theta'\rangle$ is not **1**, so it must be **2**. We have shown $B' \subseteq B$, so by symmetry we see that the two bodies are equal in this case, too. Thus Theorem 5.2 is proved. □

In the remaining types, the bodies are not necessarily equal. Any finite set (with no operations) is an E-minimal algebra of type **1**, and every subset is the body for some prime quotient. Let **S** be the four-element meet-semilattice on $\{0, a, b, 1\}$ with ordering $0 < a, b < 1$, where a and b are incomparable. It is easy to check that the sets $\{1, a\}$, $\{1, b\}$, S are all bodies for suitable prime quotients, for which **S** is minimal of type **5**.

6. Examples

Although there is a lot of information known about minimal algebras, further nontrivial properties may exist, especially in the type **2** case.

Problem 6.1. Give a complete characterization of $\langle\delta, \theta\rangle$-minimal algebras of type **2**.

What we have in mind is a result like Theorem 13.9 of [7], or Theorem 4.4, which allows one to construct examples for given purposes. What nontrivial properties must the algebra $\mathbf{C}/\tau(\mathbf{C})$ have? We present two examples to refute some possible conjectures. We call the attention of the reader to Exercise 4.37 (2) of [7] showing that a type **2** minimal algebra can have an arbitrarily long tail. We now give a more general form of that construction, due to Peter Pröhle.

Let **C** be a $\langle\delta, \theta\rangle$-minimal algebra, and **S** any meet-semilattice. Let $C^* = C \cup S$ (disjoint union), and for any function f on C, define

$$f^*(x_1, \dots, x_n) = \begin{cases} f(x_1, \dots, x_n) & \text{if } \{x_1, \dots, x_n\} \subseteq C \\ \bigwedge(\{x_1, \dots, x_n\} \cap S) & \text{otherwise.} \end{cases}$$

That is, if there is an argument outside C, then the result is the meet in the semilattice **S** of all these arguments. Define an algebra $\mathbf{C}^*$ on C^* so that its basic operations are the functions f^*, where f runs over the basic operations of **C**. Extend δ and θ to δ^* and θ^* by joining them with 0_S.

Example 6.2. The algebra $\mathbf{C}^*$ is $\langle\delta^*, \theta^*\rangle$-minimal of the same type and body as **C**.

This example works because nothing that happens in S can influence the structure of the induced algebra on **C** (since S is an ideal). We could have added any congruence of **S** to θ and δ instead of 0_S, or we could have chosen different extensions for the operations. The following example is nontrivial in this respect (a similar example is found in [8]).

Let $\mathbf{C} = \langle \{0, 1, s, t\}, +\rangle$, where the binary operation $+$ is given by

$+$	0	1	s	t
0	0	1	s	t
1	1	0	s	t
s	s	s	s	0
t	t	t	0	t

Example 6.3. The algebra $\mathbf{C}$ is $\langle 0_C, \theta\rangle$-minimal of type $\mathbf{2}$ with body $\{0, 1\}$, where θ is the congruence with single nontrivial block $\{0, 1\}$. The function $(x + y) + z$ is a pseudo-Mal'cev operation satisfying $d(s, t, 0) = 0$.

So we can come back from the tail to the body using d (but, as is necessary by Lemma 3.6, we have $d(t, t, 0) = t \in T$). Note that the only nontrivial congruence of $\mathbf{C}$ is $\theta = \tau(\mathbf{C})$, and the type of $\langle \theta, 1_C\rangle$ is $\mathbf{4}$.

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