Tolerances as congruence images
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Tolerance images

*Tolerance:* compatible, reflexive, symmetric relation. If \( \varphi : A \to B \) is a surjective homomorphism, and \( T \) is a tolerance of \( A \), then the image of \( T \) under \( \varphi \),

\[
\varphi(T) = \{(\varphi(a), \varphi(b)) : (a, b) \in A\}
\]

is a tolerance of \( B \).

In particular, the image of every congruence is a tolerance.

**Problem**

Characterize all varieties in which *every tolerance is a homomorphic image of a congruence.* **Name:** TImC.

**Motivating example**

The variety of *all lattices* has TImC (Czédli, Grätzer).

Congruence permutable varieties: every tolerance is a congruence.

**Linear identities**

*Linear identity:* every variable occurs at most once on each side.

**Theorem (Chajda, Czédli, Halaš, Lipparini)**

Every variety defined by linear identities has TImC.

**Examples**

- All algebras of a given similarity type.
- All semigroups.
- All commutative semigroups.

**Corollary**

Every tolerance is a homomorphic image of a congruence (of an algebra not necessarily in the same variety).
A Mal’tsev-like condition

Condition $M(n)$

For any pair $(f, g)$ of $2n$-ary terms such that the identity

$$f(x_0, x_0, \ldots, x_{n-1}, x_{n-1}) \approx g(x_0, x_0, \ldots, x_{n-1}, x_{n-1})$$

holds in $V$, there exists a $4n$-ary term $h$ such that the identities

$$f(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \approx h(x_0, y_0, x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_{n-1}, y_{n-1})$$

$$g(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \approx h(y_0, x_0, y_0, x_0, \ldots, y_{n-1}, x_{n-1}, y_{n-1}, x_{n-1}, y_{n-1})$$

also hold in $V$.

Pattern: $f(xx) = g(xx)$ implies $h(xyx) = f(xy)$ and $h(yxy) = g(xy)$.

Theorem (Czédli, Kiss)

A variety satisfies TImC iff it satisfies $M(n)$ for every $n \geq 1$.

Remark: No finite set of conditions $M(n)$ suffices.

Lattice varieties

Corollary

Every variety of lattices has TImC.

Proof

If $f(\ldots, x, x, \ldots) \approx g(\ldots, x, x, \ldots)$ is a lattice identity, then let

$$h(\ldots, x, y, u, v, \ldots) = f(\ldots, x \land u, y \land v, \ldots) \lor g(\ldots, y \land u, x \land v, \ldots).$$

Then we have

$$h(\ldots, x, y, x, y, \ldots) = f(\ldots, x \land x, y \land y, \ldots) \lor g(\ldots, y \land x, x \land y, \ldots),$$

which is $f(\ldots, x, y, \ldots)$, since $f$ is monotone, and similarly,

$$h(\ldots, y, x, x, y, \ldots) = f(\ldots, y \land x, x \land y, \ldots) \lor g(\ldots, x \land x, y \land y, \ldots),$$

which is $g(\ldots, x, y, \ldots)$. Thus $M(n)$ holds.

Further examples

Positive results

The following varieties satisfy $M(n)$ for all $n$ (so have TImC).

- The variety of semilattices.
- All algebras of a given similarity type (new proof).
- All varieties of unary algebras.
Negative results: Lattices are *idempotent* algebras: \( t(x, x, \ldots, x) = x \) for every term; have a *majority* term: \( m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x \).

**Example**
There exists an idempotent variety with a majority term (generated by a 3-element algebra), which fails TImC.

Rules out possible generalizations.

**Other varieties without TImC**

**Theorem**
If a congruence \( n \)-permutable variety has TImC, then it is congruence permutable.

**Proof:** by applying \( M(n) \) to the Mal’tsev condition discovered by Hagemann and Mitschke.

**Problems: discover positive and negative examples**

- Which important semigroup varieties have TImC?
- Apply \( M(n) \) to other famous Mal’tsev conditions, as above.

**Preprint**