Growth rates of solvable algebras

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Growth rate
The growth rate of a finite algebra \( A \) is the function \( d_A(n) = \text{the least size of a generating set for } A^n \).

Examples
\( A \) is a module over a ring. Then \( d_A(n) = \Theta(n) \) (linear).
Reason: The size of a basis in a vector space \( F^n \) is \( n \).

\( A \) is a Boolean algebra. Then \( d_A(n) = \Theta(\log(n)) \) (logarithmic). The same holds if \( A \) is a simple nonabelian group.
Reason: all finitary functions on \( A \) are polynomials.

\( A \) is a unary algebra. Then \( d_A(n) = 2^{\Theta(n)} \) (exponential).
Reason: The free algebras over \( A \) have polynomially bounded size.

Wiegold dichotomy
Theorem (J. Wiegold, 1974)
\( G \) is a finite group. If \( G \) has a nontrivial abelian factor group, then \( d_G \) is linear.
Otherwise (that is, if \( G \) is perfect) \( d_G \) is logarithmic.

Remarks
• If \( B \) is a homomorphic image of \( A \), then \( d_B(n) \leq d_A(n) \). So if \( G \) has an abelian factor, then \( d_G \) is at least linear.
• If \( B \) is an expansion of \( A \), then \( d_B(n) \leq d_A(n) \). The richer the structure, the smaller the growth rate.

Wiegold-dichotomy holds for Maltsev algebras (see later).

Motivating problem
What are the possible growth rates of finite algebras?
Pointed cube terms
Example: Maltsev-term \((x_1x_2^{-1}x_3\) in groups).
\[ m \begin{pmatrix} x & y & y \\ y & y & x \end{pmatrix} \approx \begin{pmatrix} x \\ x \end{pmatrix}, \]
witnesses that \(m(x_1, x_2, x_3)\) is a 3-ary, 0-pointed, 2-cube term.

If \(\Sigma\) is a set of identities in a language \(L\), then an \(L\)-term \(F(x_1, \ldots, x_m)\) is a \(p\)-pointed, \(k\)-cube term for the variety axiomatized by \(\Sigma\) if there is a \(k \times m\) matrix \(M\) consisting of variables and \(p\) distinct constant symbols, with every column of \(M\) containing a symbol different from \(x\), such that
\[ \Sigma \models F(M) \approx \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix}. \]

Growth restrictions imposed by identities
Theorem (KKSz)
Let \(A\) be an algebra with an \(m\)-ary, \(p \geq 1\)-pointed, \(k\)-cube term, with at least one constant symbol appearing in the cube identities. If \(A^{p+k-1}\) is finitely generated, then all finite powers of \(A\) are finitely generated and \(d_A(n)\) is bounded above by a polynomial of degree at most \(\log_w(m)\), where \(w = 2k/(2k - 1)\).

- There exist finite algebras with pointed cube terms whose growth rate is \(\sim\) to a polynomial of any prescribed degree.
- The growth rate of any algebra with a pointed cube term arises as the growth rate of an algebra without a pointed cube term.
- If a basic \(\Sigma\) does not entail the existence of a pointed cube term, then \(\Sigma\) imposes no restriction on growth rates.

“Basic” identity: at most one operation symbol on both sides.
General Wiegold dichotomy

If $A$ has a 0-pointed cube term, then it generates a congruence modular variety. We say that $A$ is perfect, if $[1_A, 1_A] = 1_A$ (in the sense of the modular commutator). That is, $A$ is perfect iff it has no nontrivial abelian factor algebras.

Theorem (KKSz)
Suppose that an algebra $A$ has a 0-pointed, $k$-cube term and $A^k$ is finitely generated.

- $A$ perfect $\implies d_A(n) = O(\log(n))$.
- $A$ imperfect $\implies d_A(n) = O(n)$.

Suppose that $A$ is finite.

- $A$ perfect $\implies d_A(n) = \Theta(\log(n))$.
- $A$ imperfect $\implies d_A(n) = \Theta(n)$.

The proof uses a probabilistic argument of independent interest.

Abelianness properties

By R. McKenzie and D. Hobby in tame congruence theory:

If $\alpha, \beta, \delta \in \text{Con}(A)$, then $\alpha$ centralizes $\beta$ modulo $\delta$, that is, $C(\alpha, \beta; \delta)$ holds iff for all polynomials $t$ of $A$ we have

$(\forall a \equiv_\delta b)(\forall c \equiv_\delta d) \quad t(a, c) \equiv_\delta t(a, d) \implies t(b, c) \equiv_\delta t(b, d)$.

The commutator: $[\alpha, \beta] = \bigwedge \{\delta \in \text{Con}(A) : C(\alpha, \beta; \delta) \text{ holds}\}$.

$A$ is abelian if $[1_A, 1_A] = 0_A$ (that is, $C(1_A, 1_A; 0_A)$ holds).

Homomorphic images of abelian algebras are not always abelian.

$A$ is solvable, if there is a chain of congruences $0_A = \theta_0 < \theta_1 < \ldots < \theta_n = 1_A$ such that each $\theta_i+1/\theta_i$ is abelian. Can be expressed with the commutator the same way as for groups.

Homomorphic images, direct products and subalgebras of finite solvable algebras are solvable. A finite algebra is solvable iff only the types 1 and 2 of tame congruence theory occur in it.

Nilpotence

Developed by K. Kearnes:

When $\alpha \in \text{Con}(A)$ define $\alpha^1 = [\alpha]_1 = \alpha$ and $\alpha^{k+1} = [\alpha, (\alpha)^k]$, $[\alpha]^{k+1} = [[\alpha]_k, \alpha]$.

If $[\alpha]^{n+1} = 0$, then $\alpha$ is $n$-step left nilpotent, if $[\alpha]^{n+1} = 0$, then $\alpha$ is $n$-step right nilpotent.

Right nilpotent congruences are left nilpotent in finite algebras.

Left nilpotence implies the following condition:

$C(1_A, N^2; \delta)$ holds whenever $\delta < \theta$ and $N$ is a $<\delta, \theta$-trace. $(\dagger)$

(Here $N^2$ is considered as a binary relation, and centrality is defined naturally).

This condition is still stronger than solvability.
Theorem (K. Kearnes)
Homomorphic images of finite abelian algebras are right nilpotent.

The Hamiltonian property
A is Hamiltonian: every subalgebra is a congruence block. quasi-Hamiltonian: every maximal subalgebra is a congruence-block.

Theorem (E. W. Kiss, M. Valeriote)
A locally finite variety is abelian iff it is Hamiltonian.

Wielandt: A finite group is quasi-Hamiltonian (that is, every maximal subgroup is normal) iff it is nilpotent.

Theorem (K. Kearnes)
If a finite algebra $A$ satisfies $(\dagger)$, then it is quasi-Hamiltonian. A variety generated by a finite left nilpotent algebra is quasi-Hamiltonian. Conversely, if $A^2$ is quasi-Hamiltonian, then $V(A)$ is quasi-Hamiltonian, and its finite members satisfy $(\dagger)$.

Strongly abelian algebras
An algebra $A$ is strongly abelian, if for all polynomials $t$ we have $(\forall a, b, c, d, e) \ t(a, c) = t(b, d) \Rightarrow t(e, c) = t(e, d)$.

Let $A$ be a nontrivial finite algebra and let $B$ be a nontrivial homomorphic image of $A^k$ for some $k$.

- If $B$ is strongly abelian, then $d_A(n) = 2^{\Theta(n)}$ (exponential).
- If $B$ is abelian, then $d_A(n) = \Omega(n)$ (at least linear).

This holds, because the free algebras in the first case have polynomially bounded size, and in the second case their size is in $2^{O(n)}$ by a result of J. Berman and R. McKenzie.

Each simple factoralgebra of a finite solvable algebra $A$ is either abelian or strongly abelian, so $d_A(n)$ is at least linear.
The hierarchy of abelianness properties

(1) $A$ is solvable.
(2) $A$ is (left) nilpotent.
(3) $A$ is abelian.
(4) $A$ is a subdirect product of simple abelian algebras.
(5) $A$ generates an abelian variety.

We have (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (3). No other implications hold (except the formal consequences).

We prove that stronger abelianness properties yield a closer relationship between various growth-restricting conditions.

Example: Both (5) and (4) imply that the growth rate is non-exponential if $A$ has a Maltsev term (in which case the growth rate is linear), but (2) does not.

The hierarchy of growth-restricting conditions

(i) $A$ has a Maltsev polynomial.
(ii) $A$ has a pointed cube polynomial.
(iii) $A$ is a spread of its type 2 minimal sets (see later).
(iv) $d_A(n) \in O(n)$.
(v) $d_A(n) \notin 2^{\Omega(n)}$.
(vi) $A^n$ has no nontrivial strongly abelian factor (for all $n$).

We have (i) $\Rightarrow$ (iv), (i) $\Rightarrow$ (ii) $\Rightarrow$ (v), and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi). No other implications hold for general finite algebras.

Open problem

Is the growth rate of each finite solvable $A$ linear or exponential?

True if $A$ is nilpotent; would follow from (vi) $\Rightarrow$ (iii) for solvable $A$. 

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Spreads

Definition
Let $A$ be an algebra and $U$ a collection of subsets of $A$. A subset $S \subseteq A$ is a spread with respect to $U$ if there exists a polynomial $p$ of $A$ and (not necessarily distinct) elements $U_1, \ldots, U_k \in U$ such that $p(U_1, \ldots, U_k) = S$.

Claim
If a finite algebra $A$ is a spread of a family of subsets on which the induced algebras have Maltsev polynomials (like type $2$ minimal sets), then the growth rate of $A$ is at most linear.

Theorem (KKSz)
If $A$ is a finite solvable algebra with a Maltsev polynomial, then $A$ is a spread of its type $2$ minimal sets.

Solvable algebras

Theorem (KKSz)
Let $A$ be a finite solvable algebra that has a pointed cube term. Then $d_A(n) = \Theta(n)$.

Tool used: a new characterization of solvability.

Let $A$ be an algebra and $p$ an idempotent polynomial of $A$. The translation-digraph $\text{Tr}(p)$ on $A$ has directed edges $(c, c') = (p(c, c, \ldots, c), p(c, \ldots, c, d, c, \ldots, c))$, where $c, d \in A$.

Theorem (KKSz)
A finite algebra $A$ is solvable if and only if for every neighborhood $U$ of $A$, and every idempotent polynomial $p$ of the induced algebra $A|_U$, the directed graph $\text{Tr}(p)$ is strongly connected.

Nilpotent algebras

Theorem (KKSz)
A finite left nilpotent algebra has a Maltsev polynomial iff it has a pointed cube polynomial. Hence a finite abelian algebra has a pointed cube polynomial iff it is affine (so has a Maltsev-term).

Theorem (KKSz)
If $A$ is a finite, left nilpotent algebra, and $A^{[A]}$ does not have a nontrivial strongly abelian quotient algebra, then $A$ is a spread of its type $2$ minimal sets (hence linear).

The proof uses the quasi-Hamiltonian property for the subalgebras of $A^{[A]}$. 
Abelian varieties

Theorem (KKSz)
Let $A$ be an algebra, which is a spread of subsets whose induced algebras are affine. Then the following hold.

- If $H(A^2)$ is abelian, then there is an abelian group operation on $A$ that is compatible with all operations of $A$, and preserves all congruences of $A$.
- If the variety $V(A)$ generated by $A$ is abelian, then $A$ is affine.

Examples
An 8-element quasi-affine algebra shows that in the second statement the assumption that $V(A)$ is abelian cannot be dropped.

Another 8-element abelian algebra shows that in the first statement it is not sufficient to assume only that $H(A)$ is abelian.

Semisimple algebras

Theorem (KKSz)
Let $A$ be a finite solvable algebra and $\beta$ the intersection of all maximal congruences of $A$. If the growth rate of $A/\beta$ is linear, then $A/\beta$ has a Maltsev polynomial. In particular, if $A$ is (linear, and) a direct product of simple abelian algebras, then $A$ is Maltsev.

The proof shows that $A/\beta$ is a direct product, and not just a subdirect product of simple abelian algebras.

Example
There exist a 16-element algebra that is a direct product of two, 4-element affine (hence abelian, Maltsev) algebras, has a linear growth rate, but does not have a Maltsev polynomial.
**Summary: arbitrary**

(i) A has a Maltsev polynomial.
(ii) A has a pointed cube polynomial.
(iii) A is a spread of its type 2 minimal sets.
(iv) \(d_A(n) \in O(n)\).
(v) \(d_A(n) \notin 2\Omega(n)\).
(vi) A\(^n\) has no nontrivial strongly abelian factor (for all \(n\)).

All are equivalent if A is semisimple or if \(V(A)\) is abelian.

(i) \(\implies\) (ii)

\[\Downarrow \quad \Downarrow\]

(iii) \(\implies\) (iv) \(\implies\) (v) \(\implies\) (vi).

For arbitrary finite algebras

**Summary: solvable**

(i) A has a Maltsev polynomial.
(ii) A has a pointed cube polynomial.
(iii) A is a spread of its type 2 minimal sets.
(iv) \(d_A(n) \in O(n)\).
(v) \(d_A(n) \notin 2\Omega(n)\).
(vi) A\(^n\) has no nontrivial strongly abelian factor (for all \(n\)).

All are equivalent if A is semisimple or if \(V(A)\) is abelian.

(i) \(\implies\) (ii)

\[\nparallel \quad \nparallel\]

(iii) \(\implies\) (iv) \(\implies\) (v) \(\implies\) (vi).

For finite, solvable algebras
Summary: nilpotent

(i) $A$ has a Maltsev polynomial.
(ii) $A$ has a pointed cube polynomial.
(iii) $A$ is a spread of its type 2 minimal sets.
(iv) $d_A(n) \in O(n)$.
(v) $d_A(n) \notin 2\Omega(n)$.
(vi) $A^n$ has no nontrivial strongly abelian factor (for all $n$).

All are equivalent if $A$ is semisimple or if $V(A)$ is abelian.

(i) $\iff$ (ii)

(iii) $\iff$ (iv) $\iff$ (v) $\iff$ (vi).

For finite, left nilpotent algebras

Open problems

Is there a finite algebra $A$ such that $d_A(n) \notin \Omega(n)$ and $d_A(n) \notin O(\log(n))$? That is, whose growth rate is between logarithmic and linear? Open for 2-element partial algebras, too.

Is it true that a finite algebra with a 2-sided unit for some binary term has logarithmic or linear growth? (Note that the identities $x \ast 1 = 1 \ast x = x$ show that $\ast$ is a 1-pointed 2-cube term.)

Does (ii)$\implies$(iii) hold for finite solvable algebras?

Which of the true implications (iii)$\implies$(iv)$\implies$(v)$\implies$(vi) can be reversed for finite solvable algebras? In particular, is the growth rate of a finite solvable algebra always linear or exponential?
Literature


- KKSz: *Growth rates of algebras I: pointed cube terms*. Arxiv: 1311.2352

- KKSz: *Growth rates of algebras II: Wiegold dichotomy*. Arxiv: 1311.6189

- KKSz: *Growth rates of algebras III: Solvable algebras*. Arxiv: 1311.2359