Growth rates of solvable algebras

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Keith. A Kearnes, Emil W. Kiss, Ágnes Szendrei

kearnes@euclid.colorado.edu
ewkiss@cs.elte.hu
szendrei@euclid.colorado.edu

University of Colorado Boulder, USA
Eötvös University, Budapest, Hungary
University of Colorado Boulder, USA
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$A$ is a module over a ring. Then $d_A(n) = \Theta(n)$ (linear).
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Reason: The size of a basis in a vector space $F^n$ is $n$. 
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Examples

- $\mathbf{A}$ is a module over a ring. Then $d_{\mathbf{A}}(n) = \Theta(n)$ (linear).
  - **Reason**: The size of a basis in a vector space $\mathbb{F}^n$ is $n$.

- $\mathbf{A}$ is a Boolean algebra. Then $d_{\mathbf{A}}(n) = \Theta(\log(n))$ (logarithmic).
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- $A$ is a unary algebra. Then $d_A(n) = 2^{\Theta(n)}$ (exponential).
  Reason: The free algebras over $A$ have polynomially bounded size.
Theorem (J. Wiegold, 1974)

$G$ is a finite group. If $G$ has a nontrivial abelian factor group, then $d_G$ is linear.
Wiegold dichotomy

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### Remarks

- If $B$ is a homomorphic image of $A$, then $d_B(n) \leq d_A(n)$.
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- If $B$ is an expansion of $A$, then $d_B(n) \leq d_A(n)$. 
# Wiegold dichotomy

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- If $B$ is an expansion of $A$, then $d_B(n) \leq d_A(n)$. The richer the structure, the smaller the growth rate.
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Wiegold-dichotomy holds for Maltsev algebras (see later).
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Wiegold-dichotomy holds for Maltsev algebras (see later).

**Motivating problem**

What are the possible growth rates of finite algebras?
Pointed cube terms

Example: Maltsev-term \((x_1 x_2^{-1} x_3\) in groups).
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\[
m\left(\begin{array}{ccc}
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\end{array}\right) \approx \left(\begin{array}{c}
x \\
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\end{array}\right),
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**Example:** Maltsev-term \((x_1x_2^{-1}x_3\) in groups).

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m \begin{pmatrix} x & y & y \\ y & y & x \end{pmatrix} \approx \begin{pmatrix} x \\ x \end{pmatrix},
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\[\Sigma \models F(M) \approx \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix}.\]
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m \begin{pmatrix} x & y & y \\ y & y & x \end{pmatrix} \approx \begin{pmatrix} x \\ x \end{pmatrix},
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there is a \(k \times m\) matrix \(M\) consisting of variables and \(p\) distinct constant symbols,

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there is a \(k \times m\) matrix \(M\) consisting of variables and \(p\) distinct constant symbols, with every column of \(M\) containing a symbol different from \(x\), such that

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*p*-pointed, *k*-cube term

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If \(\Sigma\) is a set of identities in a language \(\mathcal{L}\), then an \(\mathcal{L}\)-term \(F(x_1, \ldots, x_m)\) is a \(p\)-pointed, \(k\)-cube term for the variety axiomatized by \(\Sigma\) if there is a \(k \times m\) matrix \(M\) consisting of variables and \(p\) distinct constant symbols, with every column of \(M\) containing a symbol different from \(x\), such that

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witnesses that \(m(x_1, x_2, x_3)\) is a 3-ary,

If \(\Sigma\) is a set of identities in a language \(L\), then an \(L\)-term \(F(x_1, \ldots, x_m)\) is a \(p\)-pointed, \(k\)-cube term for the variety axiomatized by \(\Sigma\) if there is a \(k \times m\) matrix \(M\) consisting of variables and \(p\) distinct constant symbols, with every column of \(M\) containing a symbol different from \(x\), such that

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witnesses that \(m(x_1, x_2, x_3)\) is a 3-ary, 0-pointed, 2-cube term.

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Growth restrictions imposed by identities

Theorem (KKSz)

Let $A$ be an algebra with an $m$-ary, $p \geq 1$-pointed, $k$-cube term, with at least one constant symbol appearing in the cube identities.
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**Theorem (KKSz)**

Let $A$ be an algebra with an $m$-ary, $p \geq 1$-pointed, $k$-cube term, with at least one constant symbol appearing in the cube identities. If $A^{p+k-1}$ is finitely generated, then all finite powers of $A$ are finitely generated.
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- There exist finite algebras with pointed cube terms whose growth rate is $\sim$ to a polynomial of any prescribed degree.
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- The growth rate of any algebra with a pointed cube term arises as the growth rate of an algebra without a pointed cube term.
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- There exist finite algebras with pointed cube terms whose growth rate is $\sim$ to a polynomial of any prescribed degree.
- The growth rate of any algebra with a pointed cube term arises as the growth rate of an algebra without a pointed cube term.
- If a **basic** $\Sigma$ does not entail the existence of a pointed cube term, then $\Sigma$ imposes no restriction on growth rates. “Basic” identity: at most one operation symbol on both sides.
General Wiegold dichotomy

If $A$ has a 0-pointed cube term, then it generates a congruence modular variety.
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If $A$ has a 0-pointed cube term, then it generates a congruence modular variety. We say that $A$ is perfect, if $[1_A, 1_A] = 1_A$ (in the sense of the modular commutator).
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That is, $A$ is perfect iff it has no nontrivial abelian factor algebras.
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Suppose that an algebra \( A \) has a 0-pointed, \( k \)-cube term
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Suppose that an algebra $A$ has a 0-pointed, $k$-cube term and $A^k$ is finitely generated.
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Theorem (KKSz)

Suppose that an algebra $A$ has a 0-pointed, $k$-cube term and $A^k$ is finitely generated.

- $A$ perfect $\implies d_A(n) = O(\log(n))$. 

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- \( A \) imperfect \( \implies \) \( d_A(n) = O(n) \).
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- \( A \) imperfect \( \implies \) \( d_A(n) = O(n) \).
- Suppose that \( A \) is finite. \( A \) perfect \( \implies \) \( d_A(n) = \Theta(\log(n)) \).
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Suppose that $A$ is finite. $A$ perfect $\implies d_A(n) = \Theta(\log(n))$.
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The proof uses a probabilistic argument of independent interest.
Abelianness properties

By R. McKenzie and D. Hobby in tame congruence theory:

If $\alpha, \beta, \delta \in \text{Con}(A)$, then $\alpha$ centralizes $\beta$ modulo $\delta$, that is, $C(\alpha, \beta; \delta)$ holds iff
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The commutator: $[\alpha, \beta] = \bigwedge\{\delta \in \text{Con}(A) : C(\alpha, \beta; \delta) \text{ holds}\}$. 
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Homomorphic images of abelian algebras are not always abelian.

$A$ is solvable, if there is a chain of congruences $0_A = \theta_0 < \theta_1 < \ldots < \theta_n = 1_A$.
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$$(\forall a \equiv_{\alpha} b)(\forall c \equiv_{\beta} d) \quad t(a, c) \equiv_{\delta} t(a, d) \implies t(b, c) \equiv_{\delta} t(b, d).$$

The commutator: $[\alpha, \beta] = \bigwedge\{\delta \in \text{Con}(A) : C(\alpha, \beta; \delta) \text{ holds}\}$.

$A$ is abelian if $[1_A, 1_A] = 0_A$ (that is, $C(1_A, 1_A; 0_A)$ holds).

Homomorphic images of abelian algebras are not always abelian.

$A$ is solvable, if there is a chain of congruences

$0_A = \theta_0 < \theta_1 < \ldots < \theta_n = 1_A$ such that each $\theta_{i+1}/\theta_i$ is abelian.
Abelianness properties

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Can be expressed with the commutator the same way as for groups.
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Homomorphically, images,
## Abelianness properties

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Homomorphic images, direct products.
Abelianness properties

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Can be expressed with the commutator the same way as for groups. Homomorphomorphic images, direct products and subalgebras of finite solvable algebras are solvable.
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Can be expressed with the commutator the same way as for groups.

Homomorphic images, direct products and subalgebras of finite solvable algebras are solvable. A finite algebra is solvable iff only the types 1 and 2 of tame congruence theory occur in it.
Nilpotence

Developed by K. Kearnes:

When $\alpha \in \text{Con}(A)$ define $(\alpha)^1 = [\alpha]^1 = \alpha$
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When $\alpha \in \text{Con}(A)$ define $(\alpha)^{1} = [\alpha]^{1} = \alpha$ and
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(\alpha)^{k+1} = [\alpha, (\alpha)^{k}], \quad [\alpha]^{k+1} = [[[\alpha]^{k}, \alpha].
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If $(\alpha)^{n+1} = 0$, then $\alpha$ is $n$-step left nilpotent,
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Right nilpotent congruences are left nilpotent in finite algebras.
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Right nilpotent congruences are left nilpotent in finite algebras.
Left nilpotence implies the following condition:

$C(1_A, N^2; \delta)$ holds whenever $\delta \prec \theta$ and $N$ is a $\langle \delta, \theta \rangle$-trace.  \(\dagger\)
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(Here \( N^2 \) is considered as a binary relation, and centrality is defined naturally).
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Left nilpotence implies the following condition:

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Theorem (K. Kearnes)

Homomorphic images of finite abelian algebras are right nilpotent.
The Hamiltonian property

A is Hamiltonian: every subalgebra is a congruence block.
The Hamiltonian property

\textbf{A is Hamiltonian}: every subalgebra is a congruence block.

\textbf{quasi-Hamiltonian}: every maximal subalgebra is a congruence-block.
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Theorem (E. W. Kiss, M. Valeriote)

A locally finite variety is abelian iff it is Hamiltonian.
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Theorem (E. W. Kiss, M. Valeriote)

A locally finite variety is abelian iff it is Hamiltonian.

Wielandt: A finite group is quasi-Hamiltonian (that is, every maximal subgroup is normal) iff it is nilpotent.
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Theorem (K. Kearnes)
If a finite algebra $A$ satisfies $(†)$, then it is quasi-Hamiltonian.
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If a finite algebra $A$ satisfies (†), then it is quasi-Hamiltonian.

A variety generated by a finite left nilpotent algebra is quasi-Hamiltonian.
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A variety generated by a finite left nilpotent algebra is quasi-Hamiltonian. Conversely, if $A^2$ is quasi-Hamiltonian, then $V(A)$ is quasi-Hamiltonian,
The Hamiltonian property

\( A \) is **Hamiltonian**: every subalgebra is a congruence block.

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**Theorem (K. Kearnes)**

If a finite algebra \( A \) satisfies (†), then it is quasi-Hamiltonian. A variety generated by a finite left nilpotent algebra is quasi-Hamiltonian. Conversely, if \( A^2 \) is quasi-Hamiltonian, then \( V(A) \) is quasi-Hamiltonian, and its finite members satisfy (†).
Strongly abelian algebras

An algebra $A$ is strongly abelian, if for all polynomials $t$ we have

$$(\forall a, b, c, d, e) \quad t(a, c) = t(b, d) \implies t(e, c) = t(e, d).$$
Strongly abelian algebras

An algebra $\mathbf{A}$ is strongly abelian, if for all polynomials $t$ we have

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Let $\mathbf{A}$ be a nontrivial finite algebra and let $\mathbf{B}$ be a nontrivial homomorphic image of $\mathbf{A}^k$ for some $k$. 
Strongly abelian algebras

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Let $\mathbf{A}$ be a nontrivial finite algebra and let $\mathbf{B}$ be a nontrivial homomorphic image of $\mathbf{A}^k$ for some $k$.

- If $\mathbf{B}$ is strongly abelian, then $d_\mathbf{A}(n) = 2^{\Theta(n)}$ (exponential).
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Let $A$ be a nontrivial finite algebra and let $B$ be a nontrivial homomorphic image of $A^k$ for some $k$.

- If $B$ is strongly abelian, then $d_A(n) = 2^{\Theta(n)}$ (exponential).
- If $B$ is abelian, then $d_A(n) = \Omega(n)$ (at least linear).
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This holds, because the free algebras in the first case have polynomially bounded size,
Strongly abelian algebras

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This holds, because the free algebras in the first case have polynomially bounded size, and in the second case their size is in $2^{O(n)}$ by a result of J. Berman and R. McKenzie.
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An algebra $A$ is strongly abelian, if for all polynomials $t$ we have

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Each simple factor algebra of a finite solvable algebra $A$ is either abelian or strongly abelian,
Strongly abelian algebras

An algebra $A$ is strongly abelian, if for all polynomials $t$ we have

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Let $A$ be a nontrivial finite algebra and let $B$ be a nontrivial homomorphic image of $A^k$ for some $k$.

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Each simple factorialgebra of a finite solvable algebra $A$ is either abelian or strongly abelian, so $d_A(n)$ is at least linear.
The hierarchy of abelianness properties

(1) $A$ is solvable.
The hierarchy of abelianness properties

(1) $A$ is solvable.
(2) $A$ is (left) nilpotent.
The hierarchy of abelianness properties

(1) \( A \) is solvable.
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(3) \( A \) is abelian.
The hierarchy of abelianness properties

(1) $A$ is solvable.

(2) $A$ is (left) nilpotent.

(3) $A$ is abelian.

(4) $A$ is a subdirect product of simple abelian algebras.
The hierarchy of abelianness properties

(1) $A$ is solvable.
(2) $A$ is (left) nilpotent.
(3) $A$ is abelian.
(4) $A$ is a subdirect product of simple abelian algebras.
(5) $A$ generates an abelian variety.
The hierarchy of abelianness properties

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We have $(4) \implies (3) \implies (2) \implies (1)$
The hierarchy of abelianness properties

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We have (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (3).
The hierarchy of abelianness properties

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We have $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ and $(5) \Rightarrow (3)$.
No other implications hold (except the formal consequences).
The hierarchy of abelianness properties

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We prove that stronger abelianness properties yield
The hierarchy of abelianness properties

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We have (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (3).
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We prove that stronger abelianness properties yield a closer relationship between various growth-restricting conditions.
The hierarchy of abelianness properties

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No other implications hold (except the formal consequences).

We prove that stronger abelianness properties yield a closer relationship between various growth-restricting conditions.
Example: Both \((5)\) and \((4)\) imply that the growth rate is non-exponential iff
The hierarchy of abelianness properties

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(2) A is (left) nilpotent.
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No other implications hold (except the formal consequences).

We prove that stronger abelianness properties yield a closer relationship between various growth-restricting conditions.

Example: Both (5) and (4) imply that the growth rate is non-exponential iff A has a Maltsev term.
The hierarchy of abelianness properties

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We have \( (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \) and \( (5) \Rightarrow (3) \).
No other implications hold (except the formal consequences).

We prove that stronger abelianness properties yield a closer relationship between various growth-restricting conditions.

Example: Both (5) and (4) imply that the growth rate is non-exponential iff \( A \) has a Maltsev term (in which case the growth rate is linear),
The hierarchy of abelianness properties

(1) \(A\) is solvable.
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We have \((4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)\) and \((5) \Rightarrow (3)\).
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We prove that stronger abelianness properties yield
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**Example:** Both (5) and (4) imply that
the growth rate is non-exponential iff \(A\) has a Maltsev term
(in which case the growth rate is linear), but (2) does not.
The hierarchy of growth-restricting conditions

(i) $A$ has a Maltsev polynomial.
The hierarchy of growth-restricting conditions

(i) $A$ has a Maltsev polynomial.
(ii) $A$ has a pointed cube polynomial.
The hierarchy of growth-restricting conditions

(i) \( A \) has a Maltsev polynomial.
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(iii) \( A \) is a spread of its type 2 minimal sets (see later).
The hierarchy of growth-restricting conditions

(i) \( A \) has a Maltsev polynomial.
(ii) \( A \) has a pointed cube polynomial.
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(iv) \( d_A(n) \in O(n) \).
The hierarchy of growth-restricting conditions

(i) $A$ has a Maltsev polynomial.
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(iv) $d_A(n) \in O(n)$.
(v) $d_A(n) \notin 2\Omega(n)$. 
The hierarchy of growth-restricting conditions

(i) \( A \) has a Maltsev polynomial.

(ii) \( A \) has a pointed cube polynomial.

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(iv) \( d_A(n) \in O(n) \).

(v) \( d_A(n) \notin 2\Omega(n) \).

(vi) \( A^n \) has no nontrivial strongly abelian factor (for all \( n \)).
The hierarchy of growth-restricting conditions

(i) $\mathbf{A}$ has a Maltsev polynomial.
(ii) $\mathbf{A}$ has a pointed cube polynomial.
(iii) $\mathbf{A}$ is a spread of its type 2 minimal sets (see later).
(iv) $d_\mathbf{A}(n) \in O(n)$.
(v) $d_\mathbf{A}(n) \notin 2^{\Omega(n)}$.
(vi) $\mathbf{A}^n$ has no nontrivial strongly abelian factor (for all $n$).

We have (i)$\Rightarrow$(iv),
The hierarchy of growth-restricting conditions

(i) $A$ has a Maltsev polynomial.

(ii) $A$ has a pointed cube polynomial.

(iii) $A$ is a spread of its type 2 minimal sets (see later).

(iv) $d_A(n) \in O(n)$.

(v) $d_A(n) \notin 2\Omega(n)$.

(vi) $A^n$ has no nontrivial strongly abelian factor (for all $n$).

We have (i)$\Rightarrow$(iv), (i)$\Rightarrow$(ii)$\Rightarrow$(v),
The hierarchy of growth-restricting conditions

(i) $A$ has a Maltsev polynomial.
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(iv) $d_A(n) \in O(n)$.
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No other implications hold for general finite algebras.
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Open problem
Is the growth rate of each finite solvable \( A \) linear or exponential?
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Is the growth rate of each finite solvable \( A \) linear or exponential?

True if \( A \) is nilpotent; would follow from (vi) \( \Rightarrow \) (iii) for solvable \( A \).
Spreads

**Definition**

Let $A$ be an algebra and $U$ a collection of subsets of $A$. 
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**Claim**

If a finite algebra $A$ is a spread of a family of subsets on which the induced algebras have Maltsev polynomials
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**Claim**

If a finite algebra $A$ is a spread of a family of subsets on which the induced algebras have Maltsev polynomials (like type 2 minimal sets),
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If a finite algebra $A$ is a spread of a family of subsets on which the induced algebras have Maltsev polynomials (like type 2 minimal sets), then the growth rate of $A$ is at most linear.
## Spreads

### Definition

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If a finite algebra $A$ is a spread of a family of subsets on which the induced algebras have Maltsev polynomials (like type 2 minimal sets), then the growth rate of $A$ is at most linear.

### Theorem (KKSz)

If $A$ is a finite solvable algebra with a Maltsev polynomial, then $A$ is a spread of its type 2 minimal sets.
Solvable algebras

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Let $A$ be a finite solvable algebra that has a pointed cube term.
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Let $\mathbf{A}$ be a finite solvable algebra that has a pointed cube term. Then $d_{\mathbf{A}}(n) = \Theta(n)$. 

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A finite algebra $A$ is solvable if and only if for every neighborhood $U$ of $A$,
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### Solvable algebras

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Nilpotent algebras

**Theorem (KKSz)**

A finite left nilpotent algebra has a Maltsev polynomial iff it has a pointed cube polynomial.
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A finite left nilpotent algebra has a Maltsev polynomial iff it has a pointed cube polynomial. Hence a finite abelian algebra has a pointed cube polynomial iff it is affine.
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A finite left nilpotent algebra has a Maltsev polynomial iff it has a pointed cube polynomial. Hence a finite abelian algebra has a pointed cube polynomial iff it is affine (so has a Maltsev-term).
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If $A$ is a finite, left nilpotent algebra, and $A^{|A|}$ does not have a nontrivial strongly abelian quotient algebra,
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A finite left nilpotent algebra has a Maltsev polynomial iff it has a pointed cube polynomial. Hence a finite abelian algebra has a pointed cube polynomial iff it is affine (so has a Maltsev-term).

**Theorem (KKSz)**

If $A$ is a finite, left nilpotent algebra, and $A^{[A]}$ does not have a nontrivial strongly abelian quotient algebra, then $A$ is a spread of its type 2 minimal sets (hence linear).
Nilpotent algebras

Theorem (KKSz)
A finite left nilpotent algebra has a Maltsev polynomial iff it has a pointed cube polynomial. Hence a finite abelian algebra has a pointed cube polynomial iff it is affine (so has a Maltsev-term).

Theorem (KKSz)
If $A$ is a finite, left nilpotent algebra, and $A^{|A|}$ does not have a nontrivial strongly abelian quotient algebra, then $A$ is a spread of its type 2 minimal sets (hence linear).

The proof uses the quasi-Hamiltonian property for the subalgebras of $A^{|A|}$. 
Abelian varieties

**Theorem (KKSz)**

Let $A$ be an algebra, which is a spread of subsets whose induced algebras are affine. Then the following hold.
Abelian varieties

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Let \( A \) be an algebra, which is a spread of subsets whose induced algebras are affine. Then the following hold.

- If \( H(A^2) \) is abelian, then there is an abelian group operation on \( A \) that is compatible with all operations of \( A \),
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Let $A$ be an algebra, which is a spread of subsets whose induced algebras are affine. Then the following hold.

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- If $H(A^2)$ is abelian, then there is an abelian group operation on $A$ that is compatible with all operations of $A$, and preserves all congruences of $A$.
- If the variety $V(A)$ generated by $A$ is abelian, then $A$ is affine.
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Examples

An 8-element quasi-affine algebra shows that in the second statement the assumption that $V(A)$ is abelian cannot be dropped.
Abelian varieties

Theorem (KKSz)

Let $A$ be an algebra, which is a spread of subsets whose induced algebras are affine. Then the following hold.

- If $H(A^2)$ is abelian, then there is an abelian group operation on $A$ that is compatible with all operations of $A$, and preserves all congruences of $A$.
- If the variety $V(A)$ generated by $A$ is abelian, then $A$ is affine.

Examples

An 8-element quasi-affine algebra shows that in the second statement the assumption that $V(A)$ is abelian cannot be dropped.

Another 8-element abelian algebra shows that in the first statement it is not sufficient to assume only that $H(A)$ is abelian.
Semisimple algebras

Theorem (KKSz)

Let $A$ be a finite solvable algebra and $\beta$ the intersection of all maximal congruences of $A$. 
Semisimple algebras

**Theorem (KKSz)**

Let $A$ be a finite solvable algebra and $\beta$ the intersection of all maximal congruences of $A$. If the growth rate of $A/\beta$ is linear, then $A/\beta$ has a Maltsev polynomial.
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Let $A$ be a finite solvable algebra and $\beta$ the intersection of all maximal congruences of $A$. If the growth rate of $A/\beta$ is linear, then $A/\beta$ has a Maltsev polynomial. In particular, if $A$ is (linear, and) a direct product of simple abelian algebras, then $A$ is Maltsev.
## Semisimple algebras

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The proof shows that $A/\beta$ is a direct product, and not just a subdirect product of simple abelian algebras.
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Example

There exist a 16-element algebra that is a direct product of two, 4-element affine (hence abelian, Maltsev) algebras,
Semisimple algebras

**Theorem (KKSz)**

Let \( A \) be a finite solvable algebra and \( \beta \) the intersection of all maximal congruences of \( A \). If the growth rate of \( A/\beta \) is linear, then \( A/\beta \) has a Maltsev polynomial. In particular, if \( A \) is (linear, and) a direct product of simple abelian algebras, then \( A \) is Maltsev.

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**Example**

There exist a 16-element algebra that is a direct product of two, 4-element affine (hence abelian, Maltsev) algebras, has a linear growth rate,
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**Theorem (KKSz)**

Let $A$ be a finite solvable algebra and $\beta$ the intersection of all maximal congruences of $A$. If the growth rate of $A/\beta$ is linear, then $A/\beta$ has a Maltsev polynomial. In particular, if $A$ is (linear, and) a direct product of simple abelian algebras, then $A$ is Maltsev.

The proof shows that $A/\beta$ is a direct product, and not just a subdirect product of simple abelian algebras.

**Example**

There exist a 16-element algebra that is a direct product of two, 4-element affine (hence abelian, Maltsev) algebras, has a linear growth rate, but does not have a Maltsev polynomial.
Summary: arbitrary

(i) \( A \) has a Maltsev polynomial.
(ii) \( A \) has a pointed cube polynomial.
(iii) \( A \) is a spread of its type 2 minimal sets.
(iv) \( d_A(n) \in O(n) \).
(v) \( d_A(n) \notin 2^{\Omega(n)} \).
(vi) \( A^n \) has no nontrivial strongly abelian factor (for all \( n \)).

All are equivalent if \( A \) is semisimple or if \( V(A) \) is abelian.

\[(i) \implies (ii) \]
\[
\downarrow \quad \downarrow
\]
\[
(iii) \implies (iv) \implies (v) \implies (vi).
\]

For arbitrary finite algebras
Summary: solvable

(i) $A$ has a Maltsev polynomial.
(ii) $A$ has a pointed cube polynomial.
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$(i) \implies (ii)$

$(iii) \iff (iv) \iff (v) \implies (vi)$.

For finite, solvable algebras
Summary: nilpotent

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All are equivalent if $A$ is semisimple or if $V(A)$ is abelian.

$(i) \iff (ii)$

$(iii) \iff (iv) \iff (v) \iff (vi)$.

For finite, left nilpotent algebras
Open problems

Is there a finite algebra $A$ such that $d_A(n) \notin \Omega(n)$ and $d_A(n) \notin O(\log(n))$? That is, whose growth rate is between logarithmic and linear?
Open problems

Is there a finite algebra $A$ such that $d_A(n) \notin \Omega(n)$ and $d_A(n) \notin O(\log(n))$? That is, whose growth rate is between logarithmic and linear? Open for 2-element partial algebras, too.
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Is it true that a finite algebra with a 2-sided unit for some binary term has logarithmic or linear growth?
Open problems

Is there a finite algebra $A$ such that $d_A(n) \notin \Omega(n)$ and $d_A(n) \notin O(\log(n))$? That is, whose growth rate is between logarithmic and linear? Open for 2-element partial algebras, too.

Is it true that a finite algebra with a 2-sided unit for some binary term has logarithmic or linear growth? (Note that the identities $x \ast 1 = 1 \ast x = x$ show that $\ast$ is a 1-pointed 2-cube term.)
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Is it true that a finite algebra with a 2-sided unit for some binary term has logarithmic or linear growth? (Note that the identities $x * 1 = 1 * x = x$ show that $*$ is a 1-pointed 2-cube term.)

Does (ii)$\Rightarrow$(iii) hold for finite solvable algebras?
Open problems

Is there a finite algebra $A$ such that $d_A(n) \not\in \Omega(n)$ and $d_A(n) \not\in O(\log(n))$? That is, whose growth rate is between logarithmic and linear? Open for 2-element partial algebras, too.

Is it true that a finite algebra with a 2-sided unit for some binary term has logarithmic or linear growth? (Note that the identities $x \ast 1 = 1 \ast x = x$ show that $\ast$ is a 1-pointed 2-cube term.)

Does (ii) $\Rightarrow$ (iii) hold for finite solvable algebras?

Which of the true implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) can be reversed for finite solvable algebras?
Open problems

Is there a finite algebra \( A \) such that \( d_A(n) \notin \Omega(n) \) and \( d_A(n) \notin O(\log(n)) \)? That is, whose growth rate is between logarithmic and linear? Open for 2-element partial algebras, too.

Is it true that a finite algebra with a 2-sided unit for some binary term has logarithmic or linear growth? (Note that the identities \( x \star 1 = 1 \star x = x \) show that \( \star \) is a 1-pointed 2-cube term.)

Does (ii) \( \Rightarrow \) (iii) hold for finite solvable algebras?

Which of the true implications (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v) \( \Rightarrow \) (vi) can be reversed for finite solvable algebras? In particular, is the growth rate of a finite solvable algebra always linear or exponential?
Literature


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- KKSz: *Growth rates of algebras I: pointed cube terms*. Arxiv: 1311.2352
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- KKSz: *Growth rates of algebras I: pointed cube terms*. Arxiv: 1311.2352

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- KKSz: *Growth rates of algebras III: Solvable algebras*. Arxiv: 1311.2359
Eötvös University, Faculty of Natural Sciences

Thank you for your attention.