Three remarks on the modular commutator

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Dedicated to the memory of András Huhn

Abstract. First a problem of Ralph McKenzie is answered by proving that in a finitely directly representable variety every directly indecomposable algebra must be finite. Then we show that there is no local proof of the fundamental theorem of Abelian algebras nor of H. P. Gumm's permutability results. This part may also be of interest for those investigating non-modular Abelian algebras. Finally we provide a Gumm type-characterization of the situation when two *not necessarily comparable* congruences centralize each other. In doing this, we introduce a four variable version of the difference term in every modular variety. A "two-terms condition" is also investigated.

We assume that the reader is familiar with the basics of commutator theory, the two main introductory references are R. Freese and R. McKenzie [3] and H. P. Gumm [4]. Joins in lattices are denoted by $+$, meets by $\cdot$ or by juxtaposition.

1. Infinite directly indecomposable algebras

If a locally finite variety $\mathcal{V}$ has only finitely many directly indecomposable algebras among its *finite* members, then it has a very nice structure. Such varieties are called *finitely directly representable*, or FDR for short.

Theorem 1.1. *(R. McKenzie [8]) Let $\mathcal{V}$ be a FDR variety. Then $\mathcal{V}$ is congruence permutable, every finite directly indecomposable member of $\mathcal{V}$ is either simple or Abelian, and $\mathcal{V}$ satisfies the commutator identity $[x, y] = x \cdot y \cdot [1, 1]$ (called C2). Every subdirectly irreducible algebra of $\mathcal{V}$ is finite.*

Hence, every finite algebra in a FDR variety can be decomposed into a direct product of neutral simple algebras and an Abelian algebra. The question how the "Abelian part" looks like can be reduced to a problem about finite rings, which has been investigated earlier by ring theorists (see [8] for details). Therefore we can say that the structure of finite algebras in FDR varieties is well understood modulo ring theory. However, no result has been obtained so far about the structure of infinite algebras in such varieties.

Problem 1.2. *(R. McKenzie [8], 6.7) What are the conditions on a FDR variety in order that it have no infinite directly indecomposable members?*

The answer is this.

THEOREM 1.3. *In a FDR variety there are no infinite, directly indecomposable algebras.*

The question about infinite directly indecomposable *Abelian* algebras has been solved earlier by ring theorists. The author is indebted to István Ágoston for calling his attention to the following reference.

THEOREM 1.4. *(see H. Tachikawa [11], Corollary 9.5) Let A be a ring of finite representation type. Then every indecomposable left A-module is finitely generated, and every left A-module is a direct sum of indecomposable A-modules.*

Thus, in order to prove Theorem 1.3, it is sufficient to deal with non-Abelian, infinite, directly indecomposable algebras. First we establish discriminator polynomials in our variety. The following lemma, together with its proof, is a straightforward generalization of the paper of S. Burris [1] (and, actually, of well-known techniques dealing with discriminator varieties). Recall that the normal transform function on a set S is the four variable function $n(x, y, u, v)$ defined to be u if $x = y$ and v if $x \neq y$.

LEMMA 1.5. *Let $\mathcal{V}$ be a locally finite congruence permutable variety and k a nonnegative integer. Then there exists a term $n(x, y, u, v, z_1, \ldots, z_k)$ of $\mathcal{V}$ such that $n(x, x, u, v, \bar{z}) = u$ is an identity of $\mathcal{V}$, and for every non-Abelian simple algebra S of $\mathcal{V}$, and any system $s_1, \ldots, s_k$ of generators of S, $n(x, y, u, v, s_1, \ldots, s_k)$ is the normal transform of S.*

PROOF: Consider the set $\mathcal{M}$ of all maximal congruences η of the free algebra $F = F_{\mathcal{V}}(x, y, u, v, \bar{z})$ for which F/η is non-Abelian and $(x, y) \notin \eta$, and define γ to be the intersection of all elements of this set $\mathcal{M}$. Then F/γ is a subdirect product of neutral simple algebras, hence its congruence lattice is Boolean (Herrmann-Hagemann [5]). None of the coatoms contain the pair $(x/\gamma, y/\gamma)$, so $\mathrm{Cg}_{F/\gamma}(x/\gamma, y/\gamma) = 1_{F/\gamma}$, and thus $\gamma + \mathrm{Cg}_F(x, y) = 1_F$. Therefore there exists an element n of F with $u \mathrm{Cg}(x, y) n \gamma v$. It is routine to see that this term satisfies the conditions. ∎

Let $\mathcal{V}$ be a locally finite congruence permutable variety, $A \in \mathcal{V}$, and S a finite non-Abelian simple homomorphic image of A. Define

$$\mathcal{F} = \{\theta \in \mathrm{Con}(A) \mid A/\theta \text{ is isomorphic to a diagonal subdirect power of } S\}.$$

By a diagonal subdirect power of S we mean a subdirect product of copies of S that contains the diagonal, that is, the set of all constant functions.

LEMMA 1.6. *If $\theta \in \mathcal{F}$ and $a, b \in A$ such that $\mathrm{Cg}_A(a, b) \cdot \theta = 0$, then $\mathrm{Cg}_A(a, b)$ has a complement in $\mathrm{Con}(A)$.*

PROOF: It is sufficient to find a complement of $\mathrm{Cg}_{A/\theta}(a, b)$ in $\mathrm{Con}(A/\theta)$, because the coimage of this congruence will then be a complement of $\mathrm{Cg}_A(a, b)$ (as $\mathrm{Cg}_A(a, b) \cdot \theta = 0$). So we may assume that A is a diagonal subdirect power of S, with projection kernels η_i. Define γ to be the intersection of all η_i that do not contain (a, b). Then $\mathrm{Cg}(a, b) \cdot \gamma = 0$, and we have to show that $\mathrm{Cg}(a, b) + \gamma = 1$.

Select k to be the size of S, and apply Lemma 1.5 with this k. Substitute the diagonal elements into the term obtained. The result is a polynomial $n(x, y, u, v)$ of A that is the normal transform in every component. Now let $c, d \in A$ be arbitrary, and consider

$e = n(a,b,c,d)$. Since $n(x,x,u,v) = u$ is an identity of $\mathcal{V}$, $(c,e) \in \mathrm{Cg}(a,b)$. On the other hand, if $(a,b) \notin \eta_i$, then in the i-th component we have $n(a_i,b_i,c_i,d_i) = d_i$, thus $(e,d) \in \eta_i$. Therefore $(e,d) \in \gamma$, showing $(c,d) \in \mathrm{Cg}(a,b) + \gamma$. ∎

Thus we obtained a direct product decomposition of A (because of congruence permutability). Now assume further that S is maximal in the sense that every subdirectly irreducible homomorphic image of A is either Abelian or has cardinality not bigger than that of S. Write A as a subdirect product of subdirectly irreducible algebras S_i $(i \in I)$, with projection kernels η_i and $S = S_j$.

LEMMA 1.7. *Suppose that $B \cong S \times T$ is a subalgebra of A with projection kernels σ and τ such that $\sigma = \eta_j \restriction B$. Define $\theta = \bigwedge\{\eta_i \mid \eta_i \restriction B = \sigma\}$. Then $\theta \in \mathcal{F}$ and if $(a,b) \in \tau$ then $\mathrm{Cg}_A(a,b) \cdot \theta = 0_A$.*

PROOF: Since $B/\sigma \cong S$ is a neutral algebra, σ is neutral in $\mathrm{Con}(B)$. It is also a maximal congruence. Hence for every i we have $\eta_i \restriction B = (\eta_i \restriction B + \sigma)(\eta_i \restriction B + \tau)$, thus either $\eta_i \restriction B \le \sigma$ or $\eta_i \restriction B \ge \tau$.

In the first case we have $S \in \mathbf{HS}(S_i)$. So S_i is not Abelian, and by the maximality of S we have that $S_i \cong S$, B projects onto S_i, and $\eta_i \restriction B = \sigma$. Therefore θ is the intersection of exactly these congruences η_i. This yields a representation of A/θ as a subdirect product of copies of S, which contains $B/\sigma \cong S$ as its diagonal. Hence $\theta \in \mathcal{F}$.

In the second case we have $(a,b) \in \tau \le \eta_i \restriction B$. Hence $\mathrm{Cg}_A(a,b)$ is below the meet of all these congruences η_i. So intersecting it by θ, which is the intersection of all other projection kernels, we get 0_A. ∎

To prove Theorem 1.3, let $\mathcal{V}$ be a FDR variety and A a directly indecomposable algebra of $\mathcal{V}$. Write A as a subdirect product of subdirectly irreducible algebras S_i $(i \in I)$, with projection kernels η_i. Every subdirectly irreducible algebra in $\mathcal{V}$ is either Abelian, or a finite simple algebra, by Theorem 1.1. So if A is not Abelian, then there exists a non-Abelian simple factor among the S_i. Let $S = S_j$ be one of these of maximal size. Such an S exists, since our variety contains only a finite number of finite non-Abelian simple algebras, by Theorem 1.1.

Since we have a subdirect product, and $\mathcal{V}$ is locally finite, there exists a finitely generated, hence finite subalgebra B of A that still projects *onto* the j-th component. Since $B \in \mathbf{SP}\{S_i\}$, every neutral simple factor of B is in $\mathbf{HS}\{S_i\}$, by Jónsson's Lemma, and hence S is maximal among these. Theorem 1.1 yields that B is a direct product of neutral simple algebras and an Abelian algebra. Since $B/(\eta_j \restriction B) \cong S$, by Jónsson's Lemma and the maximality of S we see that $B \cong S \times T$, where the first projection kernel is $\sigma = \eta_j \restriction B$.

Thus we arrived at the situation in Lemma 1.7. Since S is nontrivial, there exists $(a,b) \in \tau$ with $a \ne b$. Then $\mathrm{Cg}_A(a,b) \cdot \theta = 0$, and hence for any $(a',b') \in \mathrm{Cg}_A(a,b)$ we have $\mathrm{Cg}_A(a',b') \cdot \theta = 0$ also. Lemma 1.6 then shows that for every such pair, $A/\mathrm{Cg}(a',b')$ is a direct factor of A. So if A is directly indecomposable, then $\mathrm{Cg}_A(a',b') = 1$ whenever $a' \ne b'$. In particular, $\mathrm{Cg}(a,b) = 1$ and hence A is simple. Therefore A is finite (and is actually isomorphic to S). ∎

Remarks. 1. The following two examples show that it is not easy to control the size of the direct factors of A obtained in the previous proof. If A is a Boolean algebra with no atoms, then it cannot have a finite direct factor. On the other hand, the proof of the main result in E. W. Kiss [7] produces infinite algebras in FDR varieties such that in every direct decomposition, one of the factors must be finite.

2. The previous proof produces lots of direct decompositions for infinite algebras. Therefore we believe that some kind of Boolean product decomposition exists for all infinite algebras in any FDR variety. The following fact strengthens this belief even further.

LEMMA 1.8. *If $\mathcal{V}$ is locally finite, congruence permutable, and S is a simple non-Abelian factor of A, then the set $\mathcal{F}$ of congruences defined above is a meet-subsemilattice of* $\mathrm{Con}(A)$.

PROOF: We have to construct a diagonal into a subdirect product C of two diagonal subdirect powers C_1 and C_2 of S. We can pick $|S|+|S|$ many pairs in C so that these will project to the diagonals in the first and second components, respectively. These pairs generate a finite subalgebra B of C. Projecting B to any of the components in the decomposition of C_1 or C_2, we get the whole of S, and hence B is a subdirect subalgebra in this decomposition of C. So it suffices to prove that if a subdirect power C of S contains a finite subdirect subalgebra B, then it contains one that is isomorphic to S.

As B is finite, it is isomorphic to S^m for some m, since we are in a congruence permutable variety. Let D be the diagonal of B in this decomposition. Now S is neutral simple, so $\mathrm{Con}(B)$ is Boolean. Hence each maximal congruence of B restricted to D becomes zero. So the projection kernels of the decomposition of B given by that of C also restrict trivially to D. Therefore D is still a subdirect subalgebra of C, and it is isomorphic to S. ∎

We encourage the reader to reformulate the definition of $\mathcal{F}$ in the following way. Consider the subdirect decomposition of A given above, and all subsets X of I such that $\prod\{S_i \mid i \in X\}$ has a subdirect subalgebra isomorphic to S. This family, together with the empty set, forms an ideal of subsets of I, and one has the patchwork property along all coequalizers that belong to this ideal (Lemma 1.6).

PROBLEM 1.9. *Develop a structure theory for infinite algebras in FDR varieties. Is it true that every such algebra over its center is a Boolean product of finite algebras? Does there always exist a Davey-Werner type duality (see [2]) for such varieties?*

2. LOCAL MODULARITY

The proof of both the fundamental theorem of Abelian algebras and of H. P. Gumm's permutability results depends on the difference term that is composed from the Day terms in a very nontrivial way. Therefore one can ask whether it is possible to avoid these calculations involving terms, when establishing the above results. Unfortunately, this is not the case, as shown by the theorem below. Recall that a *quasi affine algebra* is a subalgebra of a reduct of a module over an associative ring.

THEOREM 2.1. *There exists a quasi affine algebra A such that all algebras in $\mathbf{HSP}_f(A)$ have modular congruence lattices and satisfy the term condition, but A has no Mal'cev term, moreover, $A \times A$ is not congruence permutable.*

Before presenting our example, we prove some easy facts about the relationship between the concepts occurring in this theorem, which may be interesting for those investigating non-modular Abelian algebras (Lemma 2.4 in particular). Let us recall the diagram of the *shifting lemma* in an algebra A:

$$\psi\Bigg(\;\begin{array}{ccc} x & \overset{\beta}{\rule{6em}{0.4pt}} & u \\ \Big|\,\alpha & & \alpha\,\Big| \\ y & \overset{\beta}{\rule{6em}{0.4pt}} & v \end{array}$$

implies $(u,v)\in\psi$, provided $\alpha,\beta,\psi\in\mathrm{Con}(A)$ and $\alpha\beta\le\psi$. If we require this implication for every subalgebra β of $A\times A$ containing the diagonal, and not just for congruences, we obtain the *shifting principle.* If $\mathrm{Con}(A)$ is modular, then A satisfies the shifting lemma. Conversely, if A satisfies the shifting principle, then $\mathrm{Con}(A)$ is modular (Gumm [4], Lemma 3.2).

Lemma 2.2. *If* $\mathbf{S}(A\times A)$ *satisfies the shifting lemma, then* A *satisfies the shifting principle, hence* $\mathrm{Con}(A)$ *is modular.*

Proof: Consider the situation

$$\psi\Bigg(\;\begin{array}{ccc} x & \overset{\Lambda}{\rule{6em}{0.4pt}} & u \\ \Big|\,\alpha & & \alpha\,\Big| \\ y & \overset{\Lambda}{\rule{6em}{0.4pt}} & v \end{array}$$

where Λ is a reflexive subalgebra of $A\times A$, and $\alpha\cap\Lambda\subseteq\psi$. We have to show that $u\psi v$. One may assume that $\psi\le\alpha$ (replace ψ by $\alpha\psi$). The equality $(\alpha\times\alpha)\cdot(1\times 0)=\alpha\times 0$ holds in every subalgebra of $A\times A$, in particular in Λ. By the shifting lemma in Λ, we obtain, via the diagram

$$[(\alpha\times\alpha)\cdot(1\times 0)]+(\psi\times\alpha)\Bigg(\;\begin{array}{ccc} xu & \overset{1\times 0}{\rule{6em}{0.4pt}} & uu \\ \Big|\,\alpha\times\alpha & & \alpha\times\alpha\,\Big| \\ yv & \overset{1\times 0}{\rule{6em}{0.4pt}} & vv \end{array}$$

that $(uu,vv)\in[(\alpha\times\alpha)\cdot(1\times 0)]+(\psi\times\alpha)=(\alpha\times 0)+(\psi\times\alpha)$ (where $uu,xu,\ldots$ abbreviate the ordered pairs $(u,u),(x,u),\ldots$). So there exists a chain

$$\begin{array}{ccccccccc} u & \overset{\alpha}{\rule{4em}{0.4pt}} & a_1 & \overset{\psi}{\rule{4em}{0.4pt}} & a_2 & \overset{\alpha}{\rule{4em}{0.4pt}} & a_3\ \ldots & & v \\ \Lambda\,\Big| & & \Lambda\,\Big| & & \Lambda\,\Big| & & \Lambda\,\Big| & & \Lambda\,\Big| \\ u & \overset{=}{\rule{4em}{0.4pt}} & b_1 & \overset{\alpha}{\rule{4em}{0.4pt}} & b_2 & \overset{=}{\rule{4em}{0.4pt}} & b_3\ \ldots & & v \end{array}$$

between uu and vv. Since $uu\in\psi\le\alpha$, we have $a_1b_1\in\alpha\cap\Lambda\subseteq\psi\le\alpha$. Hence $a_2b_2\in\alpha\cap\Lambda\subseteq\psi\le\alpha$, and so on. Thus, every pair $a_ib_i\in\psi\le\alpha$. Now consider the top row of the above diagram. We have ψ everywhere, since, say, $a_2\psi b_2=b_3\psi a_3$. Therefore $uv\in\psi$ as desired. ∎

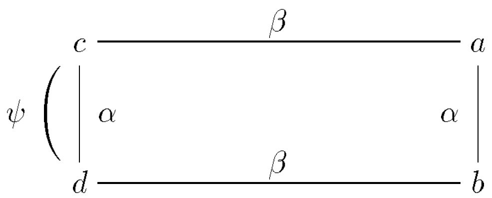
To ensure the shifting lemma, it is sufficient to have Day terms locally.

Lemma 2.3. *Let A be an algebra and $a, b, c, d \in A$. If there exist polynomials $m_0, \ldots, m_t$ of A such that*

$$\begin{array}{lll} m_0(a,b,c,d) = a & m_t(a,b,c,d) = d & \\ m_i(a,a,c,c) = a \quad \forall i & m_i(a,b,a,b) = m_{i+1}(a,b,a,b) & i \text{ odd} \\ & m_i(a,b,c,c) = m_{i+1}(a,b,c,c) & i \text{ even}, \end{array}$$

then the shifting lemma holds for the elements c, d, a, b (and all congruences α, β, ψ) in A.

Proof: Define $e_i = m_i(a,b,c,d)$, and assume

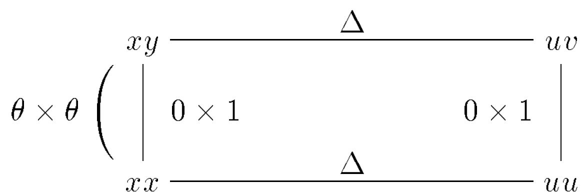


Then by the equations above, $e_0 = a$, $e_t = b$, $(e_i, a) \in \alpha$ for every i, $(e_i, e_{i+1}) \in \beta$ for every odd i, and $(e_i, e_{i+1}) \in \psi$ for every even i. Therefore $(a,b) \in \alpha \cdot \beta + \psi$. ∎

One can derive the shifting principle, too, from the existence of local Day terms, but the extra equation $m_i(a,a,a,a) = a$ for every i is needed (see the proof of Theorem 3.5 in Gumm [4]). After clarifying the local relationship between the shifting lemma, the shifting principle, and the Day terms, we investigate the term condition. Recall that TC is preserved by subalgebras and direct products, but not by homomorphic images in general.

Lemma 2.4. *Let A be an algebra satisfying the term condition. If $\mathrm{Con}(A \times A)$ is modular, then $\mathbf{H}(A)$ satisfies TC.*

Proof: Define $\Delta = \Delta_{1,1}$, that is, the smallest congruence of $A \times A$ collapsing the diagonal. Then the diagonal is a Δ-class by TC in A. Hence by the shifting lemma in $A \times A$, we have $\Delta \cdot (0 \times 1) = \Delta \cdot (1 \times 0) = 0$. Now let $\theta \in \mathrm{Con}(A)$. To prove TC in A/θ it is sufficient to present a congruence on $A/\theta \times A/\theta$ such that the diagonal is a class of it. Taking coimages, we have to produce a congruence Δ' on $A \times A$ containing $\theta \times \theta$ such that θ (as a subalgebra of $A \times A$) is a class of Δ'. We claim that $\Delta' = \Delta + (\theta \times \theta)$ is such a congruence. It is clear that all pairs of elements of θ are congruent modulo Δ', so we have to prove that the subalgebra θ is closed under both $\theta \times \theta$ and Δ. The first statement being obvious, assume that $xy \in \theta$ and $xy \Delta uv$. Then by the shifting lemma in $A \times A$ applied to the situation



we have $(uv, uu) \in \theta \times \theta$, hence $u \theta v$. Thus the subalgebra θ is closed under Δ, and hence θ is a congruence class of Δ' as required. ∎

LEMMA 2.5. *Let A be a subreduct of an affine algebra B such that $A \times A$ has permuting congruences. Then A is closed under $x - y + z$.*

PROOF: Define the congruence Δ of $B \times B$ by $\Delta = \{(ab, cd) \mid a - b = c - d\}$, and let Δ_A denote its restriction to $A \times A$. If $a, b, c \in A$, then $ab(1 \times 0)bb\Delta_A cc$ holds in $A \times A$. By our assumption, Δ_A and 1×0 permute. So there exists $pq \in A \times A$ such that $ab\Delta_A pq(1 \times 0)cc$. But then $q = c$, hence $p = a - b + c$ by the definition of Δ. Therefore $a - b + c = p \in A$. ∎

Now we prove Theorem 2.1. Let K be any field and R the (commutative) ring of polynomials $K[x, y, z, w_i \mid i < \omega]$. Define the endomorphism ρ_n $(n < \omega)$ of R by

$$\rho_n(r) = \begin{cases} w_{i+1} & \text{if } r \in \{w_i \mid i \geq n\}; \\ r & \text{if } r \in \{x, y, z, w_i \mid i < n\}. \end{cases}$$

Furthermore, introduce the operations m_1^n and m_2^n on R given by

$$m_1^n(p, q, r, s) = \rho_n(p) - w_n r + w_n s$$
$$m_2^n(p, q, r, s) = \rho_n(q) + (1 - w_n)r - (1 - w_n)s\,.$$

Let B be the algebra whose underlying set is that of R, and the operations are m_1^n and m_2^n $(n < \omega)$, and A the subalgebra of B generated by x, y, z. We show that A satisfies the conditions of Theorem 2.1.

Define B_n to be the underlying set of the subring of R generated by x, y, z and all w_i for which $i < n$. Then the underlying set of B is the ascending union of its subsets B_n, and ρ_n is the identity map on B_n. We prove that $\mathbf{HSP}_f(B)$ is modular (then the same holds for A as well). Since modularity is preserved by homomorphisms, by Lemma 2.2 it is sufficient to exhibit polynomials satisfying the conditions of Lemma 2.3 for any four-tuple (p, q, r, s) of B^k, for all integers k. Since the elements p, q, r, s have only finitely many coordinates, one can choose a large enough n so that all these coordinates are in B_n. Then ρ_n disappears from the definition of m_1^n and m_2^n, and we have

$$m_0^n(p, q, r, s) = p$$
$$m_1^n(p, q, r, s) = p - w_n r + w_n s$$
$$m_2^n(p, q, r, s) = q + (1 - w_n)r - (1 - w_n)s$$
$$m_3^n(p, q, r, s) = q\,.$$

It is routine to check that the equations in Lemma 2.3 (with $t = 3$) are satisfied.

So both $\mathbf{HSP}_f(B)$ and $\mathbf{HSP}_f(A)$ are modular, and hence satisfy the term condition by Lemma 2.4. To finish the proof of Theorem 2.1, it is sufficient to show, by Lemma 2.5, that the element $x - y + z$ of B is not contained in A.

LEMMA 2.6. *We have the following facts for every $n < \omega$.*

(i) $w_n \notin \rho_n(R)$.
(ii) ρ_n *is injective.*
(iii) *If* $x - y + z = \rho_n(p)$, *then* $x - y + z = p$.
(iv) *If* $x - y + z = m_1^n(p, q, r, s)$, *then* $x - y + z = p$.

(v) *If* $x - y + z = m_2^n(p,q,r,s)$, *then* $x - y + z = q$.

PROOF: Let π_n be the endomorphism of R mapping w_n to 0 and leaving all other generators fixed. Then $\pi_n \rho_n = \rho_n$, since this holds for all the generators of R, implying (i). For (ii) we observe that ρ_n is a bijection between the set of generators of R and a subset of these generators, and therefore one can easily construct its inverse, mapping $\rho_n(R)$ onto R. To see (iii) assume $x - y + z = \rho_n(p)$. Then as $\rho_n(x - y + z) = x - y + z$, we obtain that $\rho_n[(x - y + z) - p] = 0$, hence $x - y + z = p$ by (ii).

If $x - y + z = \rho_n(p) - w_n(r - s)$, then apply the endomorphism π_n defined above to both sides. We obtain $x - y + z = \rho_n(p)$, so (iv) holds by (iii). Finally, let π'_n be the endomorphism of R that maps w_n to 1, and fixes all the other generators. We still have that π'_n is the identity map on the range of ρ_n. So from $x - y + z = \rho_n(q) + (1 - w_n)(r - s)$ we get that $x - y + z = \rho_n(q)$, hence (v) holds by (iii). ∎

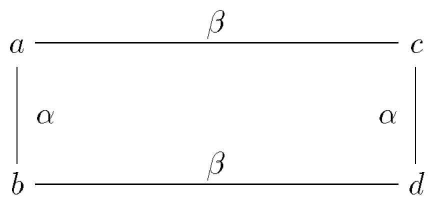
To finish the proof assume that $x - y + z \in A$. As A is generated by x, y, z, there exists a term t of A such that $t(x, y, z) = x - y + z$. Choose t of smallest possible complexity. Then (iv) and (v) show that the outermost basic operation occurring in t cannot be an m_1^n nor an m_2^n. Hence t is a projection, which is impossible, since $x - y + z \notin \{x, y, z\}$. Thus the proof of Theorem 2.1 is complete. ∎

Remark. This example was motivated by C. Herrmann's original proof of the fundamental theorem of Abelian algebras (see [6]). In that proof, the author considers an algebra A that satisfies the term condition and for which $\mathbf{HSP}_f(A)$ is modular. In the first step of the proof, assuming that A has a one element subalgebra, he embeds A into a linear algebra, so such algebras are quasi affine. In the second step it is assumed that A generates a modular variety, the difference term is constructed from the Day terms, and it is shown that it must be $x - y + z$ on A, hence A is affine. *So if we add* 0 *as a constant unary operation to our algebra B above, which makes no difference in our proofs, then the first part of C. Herrmann's argument shows that* $\mathbf{HSP}_f(A)$ *(and* $\mathbf{HSP}_f(B)$ *as well) are actually quasi affine, not only TC.* We suspect that this statement cannot be proved by extending Lemma 2.4 to state that if $A \times A$ (or $\mathbf{S}(A \times A)$) is modular, and A is quasi affine, then so is $\mathbf{H}(A)$. Similarly, we expect that the modularity of *all* finite powers of A is needed in the first part of C. Herrmann's proof.

3. From permutability to modularity

In this part we attempt to improve the technical weaponry of the commutator from a certain point of view. Our aim is to investigate the situation $[\alpha, \beta] = 0$ when α and β are not necessarily comparable congruences. First we prove a new characterization of modularity for varieties of independent interest, which is yet another way to state that modularity is permutability composed with distributivity. In this characterization only a single relation product symbol appears (as opposed to Gumm's Theorem 7.4 in [4]). Therefore it allows us to construct a single term $q(x, y, u, v)$, which will be a four variable difference term in our variety. We shall explain in detail why do we need to go up to four variables. Then we generalize, statement by statement, the basic technical assertions concerning the ternary difference term, including Gumm's characterization of $[\alpha, \beta] = 0$, but we shall not need the assumption that these two congruences are comparable. In addition, we provide a description of the congruence $\Delta_{\alpha,\beta}$ (which is a typical skew congruence in a modular variety), and a modified version of the shifting lemma. Finally we investigate briefly a "two-terms condition" characterizing $[\alpha, \beta] = 0$. We conclude the chapter by three questions about the new concepts.

First of all, we recall the results that we shall generalize, and fix the terminology. An *α-β-rectangle* (ab, cd), pictured as



is a pair in the congruence $\beta \times \beta$ of the subalgebra α of $A \times A$. The congruence $\Delta_{\alpha,\beta}$ of this algebra α, generated by all pairs (aa, cc) with $a\beta c$, therefore, consists of α-β-rectangles. By the proof of Theorem 6.14 in Gumm [4], $\Delta_{\beta,\alpha} = \{(db, ca) \mid ab\Delta_{\alpha,\beta}cd\}$, that is, one has to reflect the rectangles through their diagonal bc. Finally, $a[\alpha, \beta]b$ if and only if $aa\Delta_{\alpha,\beta}ab$ if and only if $ac\Delta_{\alpha,\beta}bc$ for some c (see [4], 6.5). We shall use these facts without reference throughout this chapter.

The fundamental theorem of Abelian algebras can be modified to state that blocks of Abelian congruences are essentially modules, and if $\alpha \leq \beta$ centralize each other, then the α-blocks lying in the same β-block are isomorphic. Let us restate the result on which the previous assertion depends. Unfortunately, there are two different versions of the ternary difference term in the literature, one in Gumm [4] called $t(x, y, z)$ and one in Freese-McKenzie [3] denoted by $d(x, y, z)$. The two are interchangable, since $d(x, y, z) = t(z, y, x)$ holds identically. The approach used below is closer in spirit to H. P. Gumm [4], however, we try to use the more recent paper, Freese-McKenzie [3], for notation and references. For the sake of consistency, we introduced some minor notational differences.

DEFINITION 3.1. *A ternary term d is called a difference term for a modular variety $\mathcal{V}$ if $d(x, x, z) = z$ is an identity of $\mathcal{V}$, and if $(a, b) \in \alpha \in \mathrm{Con}(A)$, $A \in \mathcal{V}$, then $d(a, b, b)[\alpha, \alpha]a$.*

THEOREM 3.2. *(see [3], 5.5-5.7) Every modular variety $\mathcal{V}$ has a difference term. Every difference term d in $\mathcal{V}$ satisfies the following conditions.*
(i) If $\alpha, \beta, \gamma \in \mathrm{Con}(A)$, $A \in \mathcal{V}$ and $\alpha \cdot \beta \leq \gamma$, then

with $d = d(u, u', y)$.
(ii) If $\alpha \leq \beta \in \mathrm{Con}(A)$, $A \in \mathcal{V}$, and $a\alpha b\beta c \in A$, then $(a, b) \equiv (d(a, b, c), c)$ modulo $\Delta_{\alpha,\beta}$.

(iii) *Let $\alpha \leq \beta$, $\alpha, \beta \in \mathrm{Con}(A)$, $A \in \mathcal{V}$. Then the following are necessary and sufficient conditions in order that $[\alpha, \beta] = 0$. For any basic operation (and hence any term operation) $s(x_1, \ldots, x_n)$ and elements $a_i \alpha b_i \beta c_i$ for $i = 1, \ldots, n$ we have*

$$d\left(s(\mathbf{a}), s(\mathbf{b}), s(\mathbf{c})\right) = s\left(d(a_1, b_1, c_1), \ldots, d(a_n, b_n, c_n)\right) ,$$

in other words, d and s commute on the matrix

$$\begin{matrix} a_1 & \ldots & a_n \\ b_1 & \ldots & b_n \\ c_1 & \ldots & c_n \end{matrix} \quad ,$$

and $a \alpha b$ implies $d(a, b, b) = a$, that is, d is Mal'cev on the α-blocks.

To investigate the situation $[\alpha, \beta] = 0$ without the assumption that α and β are comparable, one can use the above results to a certain extent only. Indeed, in this case we have $[\alpha + \beta, \alpha \cdot \beta] = 0$, but not conversely. In the quaternion group of order 8, the normal subgroups generated by i and j do not centralize each other, although their intersection is the center. Therefore the information $[\alpha, \beta] = 0$ is more than what Theorem 3.2 (iii) can possibly tell about this situation.

In our opinion, the reason for this phenomenon is the following. The natural objects that occur in the definition of the commutator are the α-β-rectangles. When we found all of them, we have to single out the rectangles in $\Delta_{\alpha,\beta}$. In the affine case, these are exactly those for which $a - b = c - d$, in other words, those of the form $(ab, m(abd)d)$, where m is the Mal'cev term. The rectangles in $\Delta_{\alpha,\beta}$ determine the commutator $[\alpha, \beta]$.

In the congruence permutable case plenty of such rectangles exist, since any triple $a \alpha b \beta d$ can be extended to $(ab, m(abd)d)$, where m is a Mal'cev term. In the general case we can still find enough rectangles, provided that $\alpha \leq \beta$. Indeed, comparable congruences permute, and one can complete any triple $a \alpha b \beta d$ to the (degenerate) rectangle (ab, dd). In reality, Theorem 3.2 (iii) works with such degenerate rectangles, as it will turn out below. Therefore to drop the assumption about α and β being comparable, we have to consider rectangles, not triples. Instead of blowing up triples to degenerate rectangles, we simply say: deal with all existing rectangles. The surprising fact is that even in the non-permutable case, every rectangle yields another one already in $\Delta_{\alpha,\beta}$. To find these, we must consider a four variable term, however, depending on all four vertices.

This four variable term, $q(x, y, u, v)$ seems to behave more naturally than its three variable counterpart. In a congruence distributive variety $q(x, y, u, v) = u$ can be chosen, and in an affine variety $q(x, y, u, v) = x - y + v$, independently of u. A ternary difference term in a general modular variety is $d(x, y, z) = q(x, y, z, z)$, corresponding to the degenerate rectangle discussed above. In a Mal'cev variety, $q(x, y, u, v) = m(x, y, v)$ always works. In the general case we can say that *the more affine the situation, the more q tends not to depend on its third variable.* These considerations motivate the following definition.

Definition 3.3. *A four variable term q is said to be a 4-difference term for a modular variety $\mathcal{V}$ if $q(x, y, x, y) = x$ and $q(x, x, u, u) = u$ are identities of $\mathcal{V}$, and for any two α-β-rectangles (ab, cd) and $(ab, c'd)$ of any algebra in $\mathcal{V}$ we have*

$$q(a, b, c, d)\,[\alpha, \beta]\,q(a, b, c'd)\,.$$

The two equations in this definition simply express that $q(a,b,c,d)(\alpha\cdot\beta)c$. Thus we can say that while in the congruence permutable case we can complete a partial rectangle a,b,d to $(ab, m(abd)d)$, in the general case we can modify an existing rectangle (ab,cd) to become $(ab, q(abcd)d)$, which is another rectangle (lying in $\Delta_{\alpha,\beta}$ as we shall see soon). This analogy explains why we chose the order of the arguments of q as we did.

Before formulating the generalization of Theorem 3.2, we have to prove some technical assertions as well as the promised new characterization of modularity. Congruence distributivity is characterized by $(\alpha+\gamma)(\beta+\gamma)\subseteq\alpha\beta+\gamma$. Let us see how this formula can be weakened to obtain modularity.

THEOREM 3.4. *The following are equivalent for any variety $\mathcal{V}$.*
(1) $\mathcal{V}$ is congruence modular.
(2) If $\alpha,\beta,\gamma\in\mathrm{Con}(A)$, $A\in\mathcal{V}$, then

$$(\alpha+\gamma)(\beta+\gamma)\subseteq(\alpha\beta+\gamma)\circ\alpha\,.$$

(3) If $\alpha,\beta,\gamma\in\mathrm{Con}(A)$, $A\in\mathcal{V}$ then

for some $q\in A$.

PROOF: (1) $\Rightarrow$ (2). By Theorem 8.1 of Gumm [4], for any two congruences θ and ψ, if $[\theta,\theta]$ permutes with ψ, then so does θ. Let $\psi=\alpha\beta+\gamma$ and $\theta=\alpha(\beta+\gamma)$. Then

$$[\theta,\theta]=[\alpha(\beta+\gamma),\alpha(\beta+\gamma)]\leq[\alpha,\beta+\gamma]\leq\alpha\cdot\beta+\gamma=\psi\,.$$

Since comparable congruences permute, θ permutes with ψ by our remark above. On the other hand, by modularity, we obtain

$$(\alpha+\gamma)(\beta+\gamma)=\alpha(\beta+\gamma)+\gamma=\theta+\psi\,.$$

Hence (2) holds in every modular variety indeed.

(2) $\Rightarrow$ (3). We have $ac\in\beta\leq\beta+\gamma$ and also $ac\in\alpha\circ\gamma\circ\alpha\subseteq\alpha+\gamma$. Hence $ac\in(\alpha+\gamma)(\beta+\gamma)$, so (2) yields the desired point q.

(3) $\Rightarrow$ (1). It is sufficient to prove the shifting lemma in $\mathcal{V}$. Let $\alpha,\beta,\gamma\in\mathrm{Con}(A)$ with $\alpha\beta\leq\gamma$ and consider the following situation:

We have to show that $a\gamma c$. Set $\gamma' = \gamma \cdot \beta$. Then $bd \in \gamma'$, so (3) yields q with $a\gamma' q \alpha c$. Now $aq \in \gamma' \leq \beta$, hence $q\beta c$ by transitivity. Thus $qc \in \alpha \cdot \beta \leq \gamma$ implying $a\gamma q\gamma c$. ∎

Statement (3) resembles very closely to Theorem 3.2 (i). It is stronger in the sense that q can be constructed from only four points (but it depends on all of them in general). The above proof yields some more equivalent statements, which we state separately.

COROLLARY 3.5. *The following assertions are also equivalent to those in Theorem 3.4.*

(1.5) If $\alpha, \beta, \gamma \in \mathrm{Con}(A)$, $A \in \mathcal{V}$, then

$$(\alpha + \gamma)(\beta + \gamma) = (\alpha\beta + \gamma) \circ \alpha(\beta + \gamma)\,.$$

(1.5') If $\alpha, \beta, \gamma \in \mathrm{Con}(A)$, $A \in \mathcal{V}$, then

$$(\alpha + \gamma)(\beta + \gamma) = ([\alpha, \beta] + \gamma) \circ \alpha(\beta + \gamma)\,.$$

(2') If $\alpha, \beta, \gamma \in \mathrm{Con}(A)$, $A \in \mathcal{V}$, then

$$(\alpha + \gamma)(\beta + \gamma) \subseteq ([\alpha, \beta] + \gamma) \circ \alpha\,.$$

(3') If $\alpha, \beta, \gamma \in \mathrm{Con}(A)$, $A \in \mathcal{V}$ then

for some $q \in A$.

PROOF: In the proof of (1) $\Rightarrow$ (2) above, we have actually shown that

$$[\theta, \theta] \leq \psi' = [\alpha, \beta] + \gamma\,,$$

and that $(\alpha + \gamma)(\beta + \gamma) = \theta + \gamma = \theta + \psi'$. Therefore (1) implies (1.5'). It is straightforward to check that all the other statements in the Corollary follow from (1.5') and imply (3). ∎

From now on, we fix a modular variety $\mathcal{V}$ to work in. The following statement is the essence of our considerations in this chapter.

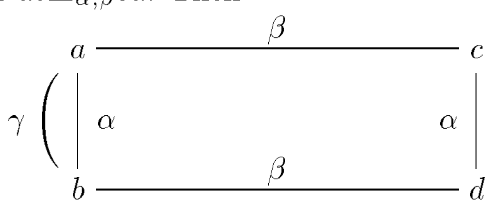
Corollary 3.6. *Let (ab, cd) be an α-β-rectangle in an algebra $A \in \mathcal{V}$. Then there exists $q \in A$ such that $(ab, qd) \in \Delta_{\alpha,\beta}$ (and $qc \in \alpha\beta$).*

Proof: Consider the following diagram in the subalgebra α of $A \times A$:

We have $(ab, cd) \in ((1 \times 0) + \Delta_{\alpha,\beta}) \cdot ((0 \times 1) + \Delta_{\alpha,\beta})$. Therefore Theorem 3.4 (2) yields that $(ab, cd) \in \Delta_{\alpha,\beta} \circ (1 \times 0)$, since $(1 \times 0) \cdot (0 \times 1) = 0$. Thus there exists a pair qd with $ab\Delta_{\alpha,\beta}qd(1 \times 0)cd$, as was to be shown. ∎

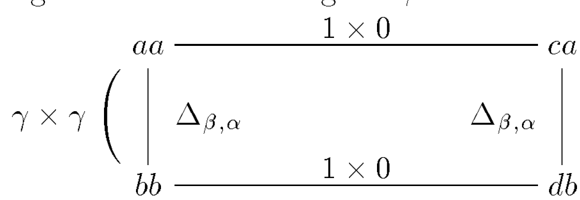
Next we give the $\Delta_{\alpha,\beta}$-version of the shifting lemma.

Proposition 3.7. *Assume $ab\Delta_{\alpha,\beta}cd$. Then*



implies $cd \in \gamma + [\alpha, \beta]$.

Proof: Apply the shifting lemma in the subalgebra β of $A \times A$ to the situation



We obtain $(ca, db) \in (\Delta_{\beta,\alpha} \cdot (1 \times 0)) + (\gamma \times \gamma)$. But $\Delta_{\beta,\alpha} \cdot (1 \times 0) \subseteq [\alpha, \beta] \times 0$. Indeed, $yx\Delta_{\beta,\alpha}vx$ yields $yv \in [\beta, \alpha]$ by the definition of the commutator. So we have $(ca, db) \in ([\alpha, \beta] \times 0) + (\gamma \times \gamma)$. Since $([\alpha, \beta] \times 0) + (\gamma \times \gamma) \leq ([\alpha, \beta] + \gamma) \times \gamma$ holds obviously, we obtain that $cd \in \gamma + [\alpha, \beta]$. ∎

Now we can state the analogue of Theorem 3.2 for the four variable difference term.

THEOREM 3.8. *Every modular variety $\mathcal{V}$ has a 4-difference term. Every 4-difference term q in $\mathcal{V}$ satisfies the following conditions.*

(i) If $\alpha, \beta, \gamma \in \mathrm{Con}(A)$, $A \in \mathcal{V}$, then

with $q = q(a, b, c, d)$, and with $q = q(d, b, c, a)$ as well.

(ii) If $\alpha, \beta \in \mathrm{Con}(A)$, $A \in \mathcal{V}$, and (ab, cd) is an α-β-rectangle, then $ab\Delta_{\alpha,\beta}q(abcd)d$, and $ab\Delta_{\alpha,\beta}cd$ if and only if $q(a, b, c, d)[\alpha, \beta]c$.

(iii) Let $\alpha, \beta \in \mathrm{Con}(A)$, $A \in \mathcal{V}$. Then the following are necessary and sufficient conditions in order that $[\alpha, \beta] = 0$. For any basic operation (and hence any term operation) $s(x_1, \ldots, x_n)$ and α-β-rectangles $(a_i b_i, c_i d_i)$ for $i = 1, \ldots, n$ we have

$$q\left(s(\mathbf{a}), s(\mathbf{b}), s(\mathbf{c}), s(\mathbf{d})\right) = s\left(q(a_1, b_1, c_1, d_1), \ldots, q(a_n, b_n, c_n, d_n)\right) ,$$

in other words, d and s commute on the matrix

$$\begin{matrix} a_1 & \ldots & a_n \\ b_1 & \ldots & b_n \\ c_1 & \ldots & c_n \\ d_1 & \ldots & d_n \end{matrix} ,$$

and q is independent of its third variable on these rectangles, that is, if (ab, cd) and $(ab, c'd)$ are α-β-rectangles, then $q(a, b, c, d) = q(a, b, c', d)$.

(iv) $d(x, y, z) = q(x, y, z, z)$ is a 3-difference term in $\mathcal{V}$ (and so is $q(z, y, z, x)$).

(v) For any two 4-difference terms q and q' of $\mathcal{V}$ and α-β-rectangle (ab, cd) in $\mathcal{V}$ we have $q(a, b, c, d)[\alpha, \beta]q'(a, b, c, d)$ and $q(a, b, c, d)[\alpha, \beta]q(d, b, c, a)$. Moreover, $q(v, y, u, x)$ is a 4-difference term also.

PROOF: To see the existence of a 4-difference term, apply Corollary 3.6 for the algebra $F_{\mathcal{V}}(x, y, u, v)$ and congruences $\theta = \mathrm{Cg}\{xy, uv\}$, $\psi = \mathrm{Cg}\{xu, yv\}$. We obtain a term q with $xy\Delta_{\theta,\psi}qv$. We show that this term $q = q(x, y, u, v)$ is a 4-difference term in $\mathcal{V}$. Indeed, as $qu \in \theta\psi$ holds in our free algebra, we obtain the equations $q(x, x, u, u) = u$ and $q(x, y, x, y) = x$ by the usual argument for proving Mal'cev conditions. Moreover, by taking homomorphic images, we see that in every algebra A of $\mathcal{V}$ and α-β-rectangle (ab, cd), we have $ab\Delta_{\alpha,\beta}q(abcd)d$ (this holds in the subalgebra generated by a, b, c, d, hence it holds in A as well). So if $(ab, c'd)$ is another rectangle, then by transitivity we have $q(abcd)d\Delta_{\alpha,\beta}q(abc'd)d$, hence $q(a, b, c, d)[\alpha, \beta]q(a, b, c', d)$ as desired. Therefore q is a 4-difference term indeed.

Next we prove (ii). First let $ab\Delta_{\alpha,\beta}cd$. By the definition of $\Delta_{\alpha,\beta}$ we know that $dd\Delta_{\alpha,\beta}bb$ (since $d\beta b$). Hence we get $q(ab, bb, cd, dd)\Delta_{\alpha,\beta}q(cd, bb, cd, bb)$, since q preserves congruences. So by the equations for q we see that $q(abcd)d\Delta_{\alpha,\beta}cd$. By the definition of the commutator, $q(a,b,c,d)[\alpha,\beta]c$.

Now let (ab,cd) be an α-β-rectangle in A. By Corollary 3.6, there exists an element $q' \in A$ with $ab\Delta_{\alpha,\beta}q'd$. Applying the result of the previous paragraph we obtain $q(abq'd)d\Delta_{\alpha,\beta}q'd$. Finally we have $q(a,b,c,d)[\alpha,\beta]q(a,b,q',d)$ by the definition of a 4-difference term, yielding $q(abcd)d\Delta_{\alpha,\beta}q(abq'd)d$. So the transitivity of $\Delta_{\alpha,\beta}$ gives (in three steps) that $ab\Delta_{\alpha,\beta}q(abcd)d$.

Finally assume that for a rectangle (ab,cd) we have $q(a,b,c,d)[\alpha,\beta]c$. Then we obtain $ab\Delta_{\alpha,\beta}q(abcd)d\Delta_{\alpha,\beta}cd$ by the result of the previous paragraph and the definition of the commutator. Therefore $ab\Delta_{\alpha,\beta}cd$, and the proof of (ii) is complete.

For (v), note that $q'(abcd)d\Delta_{\alpha,\beta}ab\Delta_{\alpha,\beta}q(abcd)d$, hence $q(a,b,c,d)[\alpha,\beta]q'(a,b,c,d)$. Set $q'(x,y,u,v) = q(v,y,u,x)$. Then the equations for q' follow directly from the equations for q. If (ab,cd) is an α-β-rectangle, then (db,ca) is a β-α-rectangle, so q' is also a 4-difference term by the commutativity of the commutator. Thus (v) holds.

It is easy to verify (iv). Indeed, let $d(x,y,z) = q(x,y,z,z)$. Then $d(x,x,z) = q(x,x,z,z) = z$, and if $x\alpha z$ then both (xz,zz) and (xz,xz) are α-α-rectangles, hence $d(x,z,z) = q(x,z,z,z)[\alpha,\alpha]q(x,z,x,z) = x$.

Next we prove (i). By (v), it makes no difference whether we choose $q = q(a,b,c,d)$ or $q = q(d,b,c,a)$, so we choose the latter one. Then (ii) shows that $db\Delta_{\beta+\gamma,\alpha}q(dbca)a$. Hence by Corollary 3.7 (the modified shifting lemma) we obtain that $q(d,b,c,a)a \in \gamma+[\beta+\gamma,\alpha] = \gamma+[\alpha,\beta]$. Since $q\alpha c$ is clear, (i) is proved.

Thus the only statement remaining is (iii). Given the rectangles (a_ib_i, c_id_i) we obtain $a_ib_i\Delta_{\alpha,\beta}q(a_ib_ic_id_i)d_i$, hence $s(\mathbf{a})s(\mathbf{b})\Delta_{\alpha,\beta}s(\ldots,q(a_ib_ic_id_i),\ldots)s(\mathbf{d})$. On the other hand, $s(\mathbf{a})s(\mathbf{b})\Delta_{\alpha,\beta}q\left(s(\mathbf{a})s(\mathbf{b})s(\mathbf{c})s(\mathbf{d})\right)s(\mathbf{d})$ by (ii), so we always have that

$$q\left(s(\mathbf{a}),s(\mathbf{b}),s(\mathbf{c}),s(\mathbf{d})\right)\;[\alpha,\beta]\,s\left(q(a_1,b_1,c_1,d_1),\ldots,q(a_n,b_n,c_n,d_n)\right).$$

This proves one direction of (iii). For the converse suppose that we have equality here, and that q is independent of its third variable on the α-β-rectangles. We have to show that $[\alpha,\beta]=0$. Define the relation Δ on the algebra α by

$$\Delta = \{(ab, q(abcd)d) \mid (ab,cd) \text{ is an } \alpha\text{-}\beta\text{-rectangle}\}.$$

We show that $\Delta = \Delta_{\alpha,\beta}$. First, Δ is compatible by the equation above. It is also reflexive, since $q(a,b,a,b) = a$. For symmetry, let $ab\Delta qd$ with $q = q(a,b,c,d)$. Since q commutes with itself on all rectangles, the matrix

$$\begin{matrix} a & b & c & d \\ b & b & d & d \\ a & b & a & b \\ b & b & b & b \end{matrix}$$

yields $q(q(a,b,c,d),d,a,b) = q(a,b,q(c,d,a,b),b)$. The right hand side of this equation equals $q(a,b,a,b) = a$, since q is independent of its third variable on rectangles. Therefore the rectangle (qd,ab) shows that $(qd,q(qdab)b) = (qd,ab) \in \Delta$, proving symmetry.

For transitivity suppose that $ab\Delta q(abcd)d$ and $q(abcd)d\Delta q(q(abcd)def)f$. Consider the following table:

$$\begin{matrix} a & b & a & b \\ b & b & b & b \\ c & d & e & f \\ d & d & f & f \end{matrix}.$$

Using the independence property of q we obtain that $q(a,b,e,f) = q(q(a,b,c,d),d,e,f)$, hence $ab\Delta q(q(abcd)def)f$ via the rectangle (ab,ef). So Δ is a congruence. The rectangle (aa,cc) for $a\beta c$ shows that $aa\Delta cc$, hence $\Delta_{\alpha,\beta} \subseteq \Delta$. On the other hand, $\Delta \subseteq \Delta_{\alpha,\beta}$ is clear by (ii). Therefore $\Delta = \Delta_{\alpha,\beta}$ indeed.

To finish the proof of $[\alpha,\beta] = 0$ suppose that $a[\alpha,\beta]b$. Then $aa\Delta_{\alpha,\beta}ab$, so there exists a rectangle (xy,uv) such that $(aa,ab) = (xy,q(xyuv)v)$, that is, $q(a,a,u,b) = a$. By the independence property, we obtain $q(a,a,u,b) = q(a,a,b,b) = b$ showing $a = b$. Therefore the proof of Theorem 3.8 is complete. ∎

Remarks. 1. It is clear from the proof above that it is property (ii) that should be used as a definition of the 4-difference term, and not the one given in Definition 3.3. We chose to proceed this way only to emphasize the analogy with the three variable case.

2. Notice that the characterization in (iii) is not symmetric in α and β, though it may seem so. When we substitute the vertices of a rectangle into q, the order of them does make a difference. If we perform the reflection $(ab,cd) \to (db,ca)$ carrying $\Delta_{\alpha,\beta}$ to $\Delta_{\beta,\alpha}$, then $q(x,y,u,v)$ is mapped to the other 4-difference term $q(v,y,u,x)$ explored in (v). If $q(x,y,u,v) = d(x,y,v)$ is globally independent of u (and so we are in a permutable variety), then this transformation yields $t(x,y,z) = d(z,y,x)$.

3. In a congruence permutable variety there may exist 4-difference terms depending on all four variables, like $q(x,y,u,v) = xy^{-1}v[u^{-1}x,u^{-1}v]$ in groups.

4. If we claim Theorem 3.8 to be a generalization of Theorem 3.2, we should be able to deduce all statements of the former result easily. Here are these arguments.

(i) We have to show that in the situation of Theorem 3.2 (i), the point $d = d(u,u',y) = q(u,u',y,y)$ works. By Theorem 3.8 (i) we have that $d' = q(u,z,y,x)$ works. But $d' = q(u,z,y,x)\beta q(u,u',y,y) = d$ (since $z\beta u'$ and $x\beta y$). On the other hand, $d'\alpha y\alpha d$, hence $dd' \in \alpha\cdot\beta \le \gamma$.

(ii) Let $a\alpha b\beta c$ with $\alpha \le \beta$. Then (ab,cc) is an α-β-rectangle, so we have $ab\Delta_{\alpha,\beta}q(abcc)c$. Since we have $d(a,b,c) = q(a,b,c,c)$ by definition, Theorem 3.2 (ii) holds.

(iii) There are two directions to prove. Suppose first that $\alpha \le \beta$ and $[\alpha,\beta] = 0$. We have to obtain the equations in Theorem 3.2 from those in Theorem 3.8. Blow up every triple (a_i,b_i,c_i) to an α-β-rectangle (a_ib_i,c_ic_i). Then the equations in Theorem 3.8 (iii) yield exactly the equations in Theorem 3.2 (iii). If $a\alpha b$, then (ab,ab) and (ab,bb) are α-β-rectangles, hence $q(a,b,b,b) = q(a,b,a,b) = a$ implies $d(a,b,b) = a$.

For the other direction we have to assume Gumm's equations in an $\alpha \le \beta$ situation, and apply Theorem 3.8 to show that $[\alpha,\beta] = 0$. It is sufficient to deduce for every α-β-rectangle (ab,uc) the identity $q(a,b,u,c) = d(a,b,c)$ from Gumm's equations, because it implies the equations in Theorem 3.8 (iii). This identity follows from the fact that the terms q and $d(x,y,z) = q(x,y,z,z)$ commute on the following matrix:

$$\begin{matrix} a & b & u & c \\ b & b & c & c \\ c & u & c & u \end{matrix}\ ,$$

hence $q(a,b,u,c) = d(q(abuc),c,c) = q(d(abc),u,u,u) = d(d(abc),u,u) = d(a,b,c)$.

The situation $[\alpha,\beta] = 0$ can also be characterized by the α-β term condition, and this characterization forms the basis to extend the commutator beyond modularity, in the framework of tame congruence theory (R. McKenzie-D. Hobby [9]). However, there is no satisfactory structure theory for Abelian varieties in general. On the other hand, it is not exactly the term condition, but affine and unary algebras, which determine Abelianness according to this theory. Therefore it seems a promising direction to look for conditions stronger than TC that are still satisfied by these two kinds of algebras, in the hope that they hold in the general Abelian case as well. One possible candidate is the concept of quasi affine algebras (see Chapter 2). They were explored in R. W. Quackenbush [10], where an implicational base of this class is given, consisting of implications that are similar to, but stronger than TC. One of these seems to deserve a separate name.

DEFINITION 3.9. *Let $\alpha,\beta \in \mathrm{Con}(A)$. The α-β two-terms condition, or α-β-TTC, is the following implication. For any two terms f and g (of not necessarily the same arity), and vectors $\mathbf{x}\alpha\mathbf{y}$, $\mathbf{u}\alpha\mathbf{v}$, $\mathbf{a}\beta\mathbf{b}$, and $\mathbf{c}\beta\mathbf{d}$ of A, the eqations*

$$\begin{aligned} f(\mathbf{x},\mathbf{a}) &= g(\mathbf{u},\mathbf{c}) \\ f(\mathbf{y},\mathbf{a}) &= g(\mathbf{v},\mathbf{c}) \\ f(\mathbf{y},\mathbf{b}) &= g(\mathbf{v},\mathbf{d}) \end{aligned}$$

imply

$$f(\mathbf{x},\mathbf{b}) = g(\mathbf{u},\mathbf{d})\,.$$

PROPOSITION 3.10. *In any algebra A, the α-β-TTC implies the α-β-TC. If A belongs to a modular variety, then they are both equivalent to $[\alpha,\beta] = 0$.*

PROOF: To show that TTC is stronger than TC assume $f(\mathbf{x},\mathbf{a}) = f(\mathbf{y},\mathbf{a})$. To prove $f(\mathbf{x},\mathbf{b}) = f(\mathbf{y},\mathbf{b})$ choose g to be the binary second projection, $c = f(\mathbf{x},\mathbf{a})$ and $d = f(\mathbf{y},\mathbf{b})$. For the converse, assume that $[\alpha,\beta] = 0$ and let q be a 4-difference term. Then by Theorem 3.8 (iii), we have

$$\begin{aligned} f(\mathbf{x},\mathbf{b}) &= q(f(\mathbf{x},\mathbf{a}), f(\mathbf{y},\mathbf{a}), f(\mathbf{x},\mathbf{b}), f(\mathbf{y},\mathbf{b})) = \\ &= q(g(\mathbf{u},\mathbf{c}), g(\mathbf{v},\mathbf{c}), g(\mathbf{u},\mathbf{d}), g(\mathbf{v},\mathbf{d})) = g(\mathbf{u},\mathbf{d})\,. \end{aligned}$$

Indeed, the middle equality follows from the independence property, and the two other ones hold since q commutes with f and g. ∎

Since this chapter is devoted partly to the analysis of the congruence $\Delta_{\alpha,\beta}$, we include a labelled diagram of the sublattice of the congruence lattice of α generated by $\Delta_{\alpha,\beta}$ and the two projection kernels. To verify this diagram is a relatively easy calculation using the machinery of Gumm [4], we leave it to the reader.

Finally we raise three problems.

PROBLEM 3.11. *Using the Day terms and a ternary difference term, is there a reasonable way to construct a 4-difference term via composition?*

For a congruence θ on a subdirect square of an algebra A, define the "product-cover" of θ to be $\overline{\theta} = ((1 \times 0) + \theta) \cdot ((0 \times 1) + \theta)$. In the case of $\theta = \Delta_{\alpha,\beta}$, we obtain $\overline{\theta} = \beta \times \beta$. Therefore in this case, θ can be described inside $\overline{\theta}$ by

$$\theta = \{(ab, cd) \in \overline{\theta} \mid q(a, b, c, d)[\alpha, \beta]c\}.$$

Since $\Delta_{\alpha,\beta}$ is, in a sense, a very typical skew congruence, we hope that the following question has a positive answer, and will lead to an understanding of skew congruences in modular varieties.

PROBLEM 3.12. *Is it possible to describe an arbitrary congruence θ inside $\overline{\theta}$ with the help of a 4-difference term?*

The starting idea of this chapter was to characterize non-comparable congruences that centralize each other. The following question suggests an alternative way to accomplish this task.

PROBLEM 3.13. *Can one assign in a natural way to every pair α, β of congruences of A a third congruence γ of an algebra in $\mathbf{V}(A)$ such that $[\alpha, \beta] = 0$ if and only if γ is Abelian?*

Acknowledgement. Most of the results in Chapter 2 and Chapter 3 of this paper were obtained during the author's visit at the University of Manitoba, Winnipeg, Canada, in the first half of 1984. The author is grateful to Professor R. W. Quackenbush for many helpful conversations concerning the topic of the paper, and for making this visit possible. Support from the NSERC of Canada is also gratefully acknowledged.

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