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LEFT AND RIGHT NILPOTENCE DEGREE ARE INDEPENDENT

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ABSTRACT. We show that for any integers $\ell, r \geq 2$ there is a finite algebra whose left nilpotence degree is ℓ and whose right nilpotence degree is r.

1. INTRODUCTION

Chapter 3 of [2] introduces an extension of the group commutator operation to arbitrary algebras. This operation is a binary operation on the set of congruences of an algebra. For congruence modular varieties the general commutator has essentially the same properties as the group commutator, but for nonmodular varieties its properties are somewhat weaker; for example, the general commutator may fail to satisfy $[\alpha, \beta] = [\beta, \alpha]$ for congruences α and β on an algebra in a nonmodular variety. This leads to (at least) two possible nilpotence concepts: when α is a congruence define $(\alpha]^1 = [\alpha)^1 = \alpha$ and

$$(\alpha]^{k+1} = [\alpha, (\alpha]^k], \quad [\alpha)^{k+1} = [[\alpha)^k, \alpha].$$

If $(\alpha]^{n+1} = 0$, then we say that α is *n-step left nilpotent*, while if $[\alpha)^{n+1} = 0$, then α is *n-step right nilpotent*. The *left nilpotence degree* of α is the smallest $n \geq 1$ such that α is n-step left nilpotent, and the *right nilpotence degree* is defined symmetrically.

A nontrivial relationship between left and right nilpotence for congruences of finite algebras was discovered in [4]:

Theorem 1.1. (Theorem 3.5 of [4]) *Let $\mathbf{A}$ be a finite algebra. The right nilpotent congruences of $\mathbf{A}$ are left nilpotent.*

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For the group commutator the converse of this statement is true; moreover the left and right nilpotence degree of any group congruence agree. But it is shown in [4] that left nilpotent congruences on arbitrary finite algebras need not be right nilpotent, while for infinite algebras neither nilpotence condition implies the other.

The purpose of this paper is to solve two problems raised in [4] concerning the independence of the left and right nilpotence degrees of congruences on finite algebras. We first solve Problem 1 of that paper:

Problem 1. Show that if β is a right nilpotent congruence on a finite algebra and $\alpha \leq \beta$, then $[\beta, \alpha] \leq [\alpha, \beta]$.

We 'solve' this problem by producing a counterexample. (Let $\beta = 1_L$ and $\alpha = \theta_{\ell-1}$ in the example from Section 2 below.) We then solve Problem 2 of [4]:

Problem 2. Find all pairs (ℓ, r) such that there exists a finite algebra $\mathbf{A}$ with a congruence θ that has left nilpotence degree ℓ and right nilpotence degree r.

Clearly $\ell = 1$ iff θ is abelian iff $r = 1$, so the solution to this problem requires one to find only the pairs (ℓ, r) for which $\ell, r \geq 2$. We will show that for any $\ell, r \geq 2$ there is a finite algebra $\mathbf{A}$ whose largest congruence $\theta = 1_A$ has nilpotence type (ℓ, r). This task is simplified by an observation from [4]: if $\mathbf{A}$ is a finite algebra with a congruence θ of nilpotence type (ℓ, r) and $\mathbf{A}'$ is a finite algebra with a congruence θ' of nilpotence type (ℓ', r'), then the nonindexed product $\mathbf{A} \otimes \mathbf{A}'$ has the product congruence $\theta \times \theta'$ which has nilpotence type equal to $(\max(\ell, \ell'), \max(r, r'))$. Thus, to show that nilpotence type (ℓ, r) can be realized it suffices to prove that nilpotence type $(\ell, 2)$ can be realized (see Section 2) and that nilpotence type $(2, r)$ can be realized (see Section 3). See [3] for earlier partial results on Problem 2.

Define the *extended* (left, right) nilpotence degree of a congruence θ to be equal to ∞ if the congruence is not (left, right) nilpotent, and to be equal to its usual nilpotence degree if it is nilpotent. The statement made about the left or right nilpotence degree of a product congruence on $\mathbf{A} \otimes \mathbf{A}'$ holds also for the extended left or right nilpotence degree if we order the set $\{1, 2, \ldots, \infty\}$ of possible degrees with $n < \infty$ for all n. Example 1 of [4] shows that there is a finite algebra whose total congruence 1_A has extended nilpotence type $(2, \infty)$, and Example 3 of [4] shows that there is an infinite algebra whose total congruence 1_A has extended nilpotence type $(\infty, 2)$. Using nonindexed products and the results of this paper, this implies that there exist infinite algebras whose total congruence has any given extended nilpotence type (ℓ, r) with $\ell, r \geq 2$. It also implies that there exist finite algebras whose total congruence has any given extended nilpotence type (ℓ, r) with $\ell, r \geq 2$ except types (∞, n), $n < \infty$, which **must** be excluded according to Theorem 1.1.

The pathology we describe in this paper can only arise in algebras that have 'very weak' operations. In any variety with a weak difference term (the existence of which is equivalent to a fairly weak idempotent Maltsev condition) the behavior of the commutator on solvable congruence intervals is the same as the behavior that one finds on solvable congruence intervals in congruence permutable varieties. In particular, the extended left nilpotence degree of any congruence equals the extended right nilpotence degree. Moreover, it follows from Lemma 3.2 of [4] that the commutator properties of solvable congruence intervals of finite algebras that omit type **1** are again well behaved, and again the extended left nilpotence degree of any congruence equals the extended right nilpotence degree. Thus, the examples we construct must include type **1**. In fact, they are both *strongly nilpotent algebras* in the sense of [5], and are *E-minimal algebras of type* **1**. E-minimal algebras are defined and described in [2]. The structure theorem for the type **1** case is given in Theorem 4.4 of [6]. The reader can easily verify using this theorem that the algebras we construct are indeed E-minimal. They are strongly nilpotent by Lemma 3.4 of [5].

We close the Introduction by recalling the precise definition (from [2]) of the general commutator operation. If $\mathbf{A}$ is an algebra with congruences α, β and δ, then we say that the relation $C(\alpha, \beta; \delta)$ *holds* if whenever $t(x, \mathbf{y})$ is an $(n+1)$-ary term of $\mathbf{A}$, $(a, b) \in \alpha$, $\mathbf{u}, \mathbf{v} \in A^n$ with $(u_i, v_i) \in \beta$, then

$$t(\underline{a}, \mathbf{u}) \equiv_\delta t(\underline{a}, \mathbf{v})$$

implies that

$$t(\underline{b}, \mathbf{u}) \equiv_\delta t(\underline{b}, \mathbf{v}).$$

(The underlined values changed without changing the δ-relation.) The *commutator* of α and β is the least congruence $[\alpha, \beta]$ such that $C(\alpha, \beta; [\alpha, \beta])$ holds.

2. The $(\ell, 2)$-example

In this section we construct an algebra $\mathbf{L}$ of nilpotence type $(\ell, 2)$ on the set $L = \{0, 1, \ldots, \ell\}$. $\mathbf{L}$ has one binary operation given by the following table:

$*$	0	1	2	$\cdots$	$\ell-1$	ℓ
0	0	0	1	$\cdots$	$\ell-2$	$\ell-1$
1	0	0	1	$\cdots$	$\ell-2$	$\ell-1$
2	0	0	1	$\cdots$	$\ell-2$	$\ell-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\ell-1$	0	0	1	$\cdots$	$\ell-2$	$\ell-1$
ℓ	$\ell-1$	$\ell-1$	$\ell-1$	$\cdots$	$\ell-1$	$\ell-1$

Thus, $x * y = \max\{0, y-1\}$ if $x \neq \ell$, and $x * y = \ell - 1$ otherwise.

Lemma 2.1. *The congruence lattice of* $\mathbf{L}$ *is the* $(\ell+1)$*-element chain*

$$0_L = \theta_0 < \theta_1 < \cdots < \theta_\ell = 1_L$$

where $\theta_i = (I \times I) \cup \Delta$ *for* $I = \{0, 1, \ldots, i\}$ *and* $\Delta = \{(x, x) \mid x \in L\}$.

Proof. To prove that the equivalence relation $\theta_i = (I \times I) \cup \Delta$ is a congruence, it suffices to show that if $(a, b) \in \theta_i$ and $c \in L$, then $(a*c, b*c), (c*a, c*b) \in \theta_i$. Only the cases where $\theta_i \neq 1_L$ are nontrivial, so we may assume that $a, b \in I \neq L$. From the multiplication table we see that under this assumption $a*c = b*c$, so $(a*c, b*c) \in \theta_i$. Moreover, if $c \neq \ell$, then $c*a, c*b \in \{0, 1, \ldots, i-1\}$ while if $c = \ell$ we get $c*a = \ell - 1 = c*b$. In either case we have $(c*a, c*b) \in \theta_i$.

To prove that we have located all congruences, it suffices to show that any principal congruence of $\mathbf{L}$ is some θ_i. For this we will argue that

(1) If $a < b$, then $\mathrm{Cg}(a, b) = \mathrm{Cg}(0, b)$, and
(2) $\mathrm{Cg}(0, b) \leq \mathrm{Cg}(0, b+1)$.

Since θ_b contains $(0, b)$ and not $(0, b+1)$, these two items will establish that the principal congruences are precisely those of the form $\mathrm{Cg}(0, b) = \theta_b$.

Item (2) follows from the fact that $\mathrm{Cg}(0, b+1)$ contains $(0*0, 0*(b+1)) = (0, b)$. Repeated use of (2) shows that $(0, a) \in \mathrm{Cg}(0, b)$ for any $a < b$. This proves part of (1): for any $a < b$ we have $(a, 0), (0, b) \in \mathrm{Cg}(0, b)$, so $\mathrm{Cg}(a, b) \leq \mathrm{Cg}(0, b)$. We prove the other inclusion by induction on b. Suppose that for all $a' < b' < b$ we have $\mathrm{Cg}(a', b') = \mathrm{Cg}(0, b')$. To prove that $\mathrm{Cg}(a, b) \geq \mathrm{Cg}(0, b)$ for all $a < b$, we may restrict ourselves to the case where $a \neq 0$. Thus $(a-1, b-1) = (0*a, 0*b) \in \mathrm{Cg}(a, b)$. By induction and (2), $(0, x) \in \mathrm{Cg}(0, b-1) = \mathrm{Cg}(a-1, b-1)$ for all $x \leq b-1$. In particular, $(0, a) \in \mathrm{Cg}(a-1, b-1) \leq \mathrm{Cg}(a, b)$. Since $\mathrm{Cg}(a, b)$ contains both $(0, a)$ and (a, b) it also contains $(0, b)$. This finishes the proof that (1) holds, and so finishes the proof of the lemma. □

Lemma 2.2. $\mathbf{L}$ *is* 2*-step right nilpotent.*

Proof. Since $\mathbf{L}/\theta_{\ell-1}$ is a 2-element algebra with a single constant operation, it is abelian. Thus $[1, 1] \leq \theta_{\ell-1}$. By proving that $[\theta_{\ell-1}, 1] = 0$ we will establish that $[[1, 1], 1] \leq [\theta_{\ell-1}, 1] = 0$, which is what the lemma claims.

Claim 2.3. *Define a binary relation* Ω *on* $\mathbf{L} \times \mathbf{L}$ *by letting* $\langle (a, b), (c, d) \rangle \in \Omega$ *iff one of the following is true:*

(*i*) $(a, b) = (c, d)$,
(*ii*) $a = b$, $c = d$ *and* $a, c < \ell$,
(*iii*) $a = c$ *and* $b, d < a < \ell$, *or*
(*iv*) $b = d$ *and* $a, c < b < \ell$.

Then Ω *is a congruence on* $\mathbf{L} \times \mathbf{L}$.

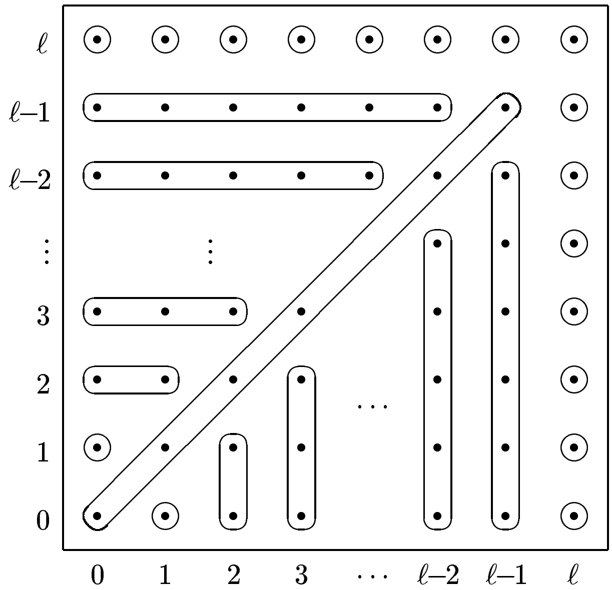

FIGURE 1. The partition associated to Ω.

It is easy to see that the relation Ω is an equivalence relation on $L \times L$ that partitions the set into the classes depicted in Figure 1.

To verify that Ω is a congruence we must show that if $(a, b) \equiv_\Omega (c, d)$ and $(e, f) \in L \times L$, then $(e, f) * (a, b) \equiv_\Omega (e, f) * (c, d)$ and $(a, b) * (e, f) \equiv_\Omega (c, d) * (e, f)$. This is trivial if we are in case (i) of the definition of Ω (the case where $(a, b) = (c, d)$). Thus we only need to consider cases (ii)–(iv). To simplify the arguments that follow, we shall use the notation x^- to mean $x - 1$ for $x > 0$ and 0 for $x = 0$.

Case (ii). In this case $a = b, c = d$ and $a, c < \ell$. We must prove that $(e, f) * (a, a) \equiv_\Omega (e, f) * (c, c)$ and $(a, a) * (e, f) \equiv_\Omega (c, c) * (e, f)$. By examining the multiplication table for $*$ we see that $(a, a) * (e, f) = (c, c) * (e, f)$ when $a, c < \ell$, so we do have $(a, a) * (e, f) \equiv_\Omega (c, c) * (e, f)$. For the other verification, if $e \neq \ell \neq f$ or $e = \ell = f$, then $(e, f) * (a, a) \in \{(a^-, a^-), (\ell^-, \ell^-)\}$ and $(e, f) * (c, c) \in \{(c^-, c^-), (\ell^-, \ell^-)\}$. In any case, item (ii) of the definition of Ω proves that $(e, f) * (a, a) \equiv_\Omega (e, f) * (c, c)$. If $e < \ell = f$, then $(e, f) * (a, a) = (a^-, \ell^-)$ while $(e, f) * (c, c) = (c^-, \ell^-)$. Thus $(e, f) * (a, a) \equiv_\Omega (e, f) * (c, c)$ by item (iv) of the definition of Ω. Similarly, if $e = \ell > f$, then $(e, f) * (a, a) \equiv_\Omega (e, f) * (c, c)$ by item (iii) of the definition of Ω.

Case (iii). In this case $a = c$ and $b, d < a < \ell$. We must prove that $(e, f) * (a, b) \equiv_\Omega (e, f) * (a, d)$ and $(a, b) * (e, f) \equiv_\Omega (a, d) * (e, f)$. For the first verification, note that if $f = \ell$ we get $(e, f) * (a, b) = (e * a, \ell^-) = (e, f) * (a, d)$, so $(e, f) * (a, b) \equiv_\Omega (e, f) * (a, d)$; but if $f \neq \ell$, then $(e, f) * (a, b) = (e * a, b^-) \equiv_\Omega (e * a, d^-) = (e, f) * (a, d)$ by item (iii) of the definition of Ω. For the second verification, $(a,b)*(e,f) = (a*e,f^-) = (a,d)*(e,f)$, so $(a,b)*(e,f) \equiv_\Omega (a,d)*(e,f)$.

Case (iv). In this case $b = d$ and $a, c < b < \ell$. This argument is similar to the previous case. The claim is proved.

It is straightforward to check that $[\theta_{\ell-1}, 1] = 0$ follows from the result stated in the previous claim (that there is some congruence Ω on $\mathbf{L}\times\mathbf{L}$ which relates (a,a) and (b,c) iff $b = c$ and $(a,b) \in \theta_{\ell-1}$). We include the details. Suppose that $p(x,\mathbf{y})$ is a term, $\mathbf{u},\mathbf{v} \in L^n$, $(r,s) \in \theta_{\ell-1}$, and

$$t(\underline{r},\mathbf{u}) = a = t(\underline{r},\mathbf{v}).$$

We wish to show that for

$$b = t(\underline{s},\mathbf{u}), \quad t(\underline{s},\mathbf{v}) = c$$

we must have $b = c$. Since $(r,s) \in \theta_{\ell-1}$, we have $(r,r) \equiv_\Omega (s,s)$ by item (ii) of the definition of Ω, and since $t((x,y),(\mathbf{u},\mathbf{v}))$ is a unary polynomial of $\mathbf{L}\times\mathbf{L}$ we get

$$(a,a) = t((r,r),(\mathbf{u},\mathbf{v})) \equiv_\Omega t((s,s),(\mathbf{u},\mathbf{v})) = (b,c).$$

By the properties of Ω, $b = c$. This establishes $C(\theta_{\ell-1},1;0)$, so $[\theta_{\ell-1},1] = 0$. $\square$

Lemma 2.4. *The left nilpotence degree of* $\mathbf{L}$ *is exactly* ℓ.

Proof. Choose $t(x,y) = x*y$, $(\ell,0) \in \theta_\ell = 1$, and $(0,i) \in \theta_i$. If we change the underlined value in the equality

$$t(\underline{\ell},0) = \underline{\ell}*0 = \ell-1 = \underline{\ell}*i = t(\underline{\ell},i),$$

from ℓ to 0 we get that

$$t(\underline{0},0) = \underline{0}*0 \equiv_{[1,\theta_i]} \underline{0}*i = t(\underline{0},i)\,.$$

Since $(0*0, 0*i) = (0, i-1)$ when $i > 0$, and $\theta_{i-1} = \mathrm{Cg}(0,i-1)$, this implies that $[1,\theta_i] \geq \theta_{i-1}$ when $i > 0$.

In Lemma 2.2 we showed that $\mathbf{L}$ is right nilpotent. By Theorem 1.1, right nilpotent algebras are left nilpotent. This implies that $[1,\theta_i] < \theta_i$ for each $i > 0$. Since the congruence lattice of $\mathbf{L}$ is a chain, it follows that $[1,\theta_i] \leq \theta_{i-1}$ for each $i > 0$. (This can be verified directly, without resorting to Theorem 1.1.) Combining this with the result of the preceding paragraph we get that $[1,\theta_i] = \theta_{i-1}$ for each $i > 0$. It follows that the descending left central series is exactly

$$1 = \theta_\ell > \theta_{\ell-1} > \cdots > \theta_1 > \theta_0 = 0,$$

and so the left nilpotence degree of $\mathbf{L}$ is ℓ. $\square$

3. The $(2, r)$-example

In this section we construct an algebra $\mathbf{R}$ of nilpotence type $(2, r)$. The construction and its verification is similar to the one presented in the previous section. The underlying set of the algebra $\mathbf{R}$ is the set R of all pairs (i, j) such that $1 \leq i \leq j \leq r$. The algebra $\mathbf{R}$ has one binary operation $*$, and its operation table has the property that all rows are the same except for the last one, corresponding to (r, r). Thus, to present the table, we have to define the function $f(y) = u * y$ for $u \neq (r, r)$, and the function $g(y) = (r, r) * y$. Let $f((i, j)) = (i - 1, j)$ if $i > 1$, and $f((1, j)) = (1, 1)$. The function g is equal to f with the exception that $g((i, i)) = (i - 1, i - 1)$ for $i > 1$. These functions are shown on Figure 2 for the case of $r = 4$ (where an unlabeled arrow $u \to v$ means that $f(u) = g(u) = v$). To make the calculations easier to follow, we shall sometimes write the pair (i, j) simply as ij (as in Figure 2).

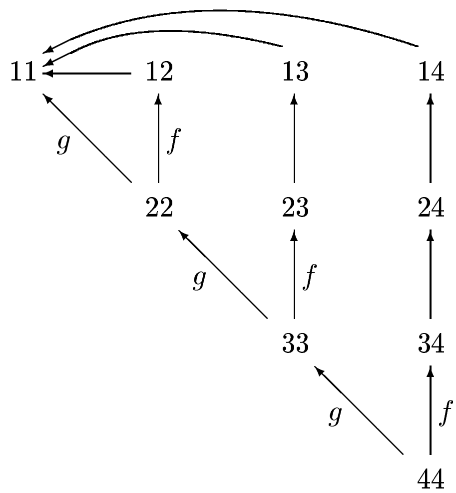

Figure 2. The mappings f and g.

The congruence lattice of $\mathbf{R}$ is not a chain in general. We shall define special congruences of $\mathbf{R}$. Let $\psi_r = 1_R$, and for $1 \leq i \leq r-1$ let ψ_i consist of the pairs (ab, ad), where $a \leq i$ or $b = d$. In Figure 2 the partition associated to ψ_i is easy to visualize: the nontrivial blocks are the first i rows. Let $S = R - \{rr\}$.

Lemma 3.1. *The relations ψ_i are congruences of $\mathbf{R}$, and they satisfy that $0_R = \psi_0 < \psi_1 < \cdots < \psi_r = 1_R$.*

Proof. To prove that the equivalence relation ψ_i is a congruence, it suffices to show that if $ab \equiv_{\psi_i} cd$ and $u \in R$, then $ab * u \equiv_{\psi_i} cd * u$ and $u * ab \equiv_{\psi_i} u * cd$. As $\psi_r = 1_R$ is obviously a congruence, we may assume that $i < r$. Then all nontrivial blocks of ψ_i are contained in the set $S = R - \{rr\}$. We may assume that ab and cd are in such a block. The rows of the operation table of $*$ corresponding to S are all equal (and are given by f). Therefore $ab*u = cd*u$, and so $ab*u \equiv_{\psi_i} cd*u$ indeed. If $u \in S$, then $u*ab = f(ab)$ and $u*cd = f(cd)$. If $u \notin S$ (that is, $u = rr$), then we get that $u * ab = g(ab)$ and $u * cd = g(cd)$. Thus we have to verify that $f(ab) \equiv_{\psi_i} f(cd)$ and $g(ab) \equiv_{\psi_i} g(cd)$.

From $ab \equiv_{\psi_i} cd$ we get that $a = c \le i$. If $a(= c) = 1$, then $f(ab) = f(cd) = g(ab) = g(cd) = 11$. If $a(= c) > 1$, then $f(ab) = (a-1, b)$, and $f(cd) = (c-1, d)$. Here $a - 1 = c - 1 \le i$, and therefore $f(ab) \equiv_{\psi_i} f(cd)$. The argument for g is similar, since g also decreases the first coordinates by one. □

Lemma 3.2. **R** *is* 2*-step left nilpotent.*

Proof. We shall prove that $[1,1] \le \psi_{r-1}$, and that $[1, \psi_{r-1}] = 0$. The first statement is clear, since in the factor modulo ψ_{r-1} the functions f and g become equal, and thus this factor is an essentially unary algebra, hence it is abelian. To prove that $[1, \psi_{r-1}] = 0$ holds we shall establish the following statement. Recall that two elements of R are ψ_{r-1}-related if and only if their first coordinates are equal.

Claim 3.3. *Denote by* $\mathbf{T}$ *the subalgebra of* $\mathbf{R} \times \mathbf{R}$ *whose universe is the set* $\{(u, v) \in R \times R \mid u \equiv_{\psi_{r-1}} v\}$*. Consider the binary relation* Ξ *on* T *defined as follows. Let* $a = a_1a_2$*,* $b = b_1b_2$*,* $c = c_1c_2$ *and* $d = d_1d_2$ *be elements of* R *such that* $(a, b), (c, d) \in T$ *(hence* $a_1 = b_1$ *and* $c_1 = d_1$*). Now* $\langle (a, b), (c, d) \rangle \in \Xi$ *iff one of the following is true:*

(*i*) $(a, b) = (c, d)$,
(*ii*) $a = b$ *and* $c = d$,
(*iii*) $a = c$ *and* $b_2, d_2 < a_2$, *or*
(*iv*) $b = d$ *and* $a_2, c_2 < b_2$.

Then Ξ *is a congruence on* $\mathbf{T}$.

Again it is easy to see that the relation Ξ is an equivalence relation on T that partitions the set into the classes depicted in Figure 3. We first establish the following facts.

(1) $(f(a), f(b)) \equiv_{\Xi} (g(a), g(b))$ for every $(a, b) \in T$.
(2) If $(a, b) \equiv_{\Xi} (c, d)$, then $(f(a), f(b)) \equiv_{\Xi} (f(c), f(d))$.
(3) If $(a, b) \equiv_{\Xi} (c, d)$, then $(g(a), g(b)) \equiv_{\Xi} (g(c), g(d))$.

The statement in (1) clearly holds if $f(a) = g(a)$ and $f(b) = g(b)$. If $f(a) \ne g(a)$, then by the definition of f and g we have that $a_1 = a_2 > 1$. From $(a, b) \in T$ we get that $a_1 = b_1$. If $b_1 = b_2$, then $a = b$, and we are done by (*ii*). If not, then for $e := a_1 - 1$ we get that $f(b) = g(b) = (e, b_2)$, $f(a) = (e, e+1)$ and $g(a) = (e, e)$. Thus the pair $((e, e+1), (e, b_2))$ is Ξ-related to $((e, e), (e, b_2))$ by (*iv*), since $e < e + 1 = b_1 < b_2$. The case when $f(b) \ne g(b)$ follows by switching a and b. Thus (1) is proved.

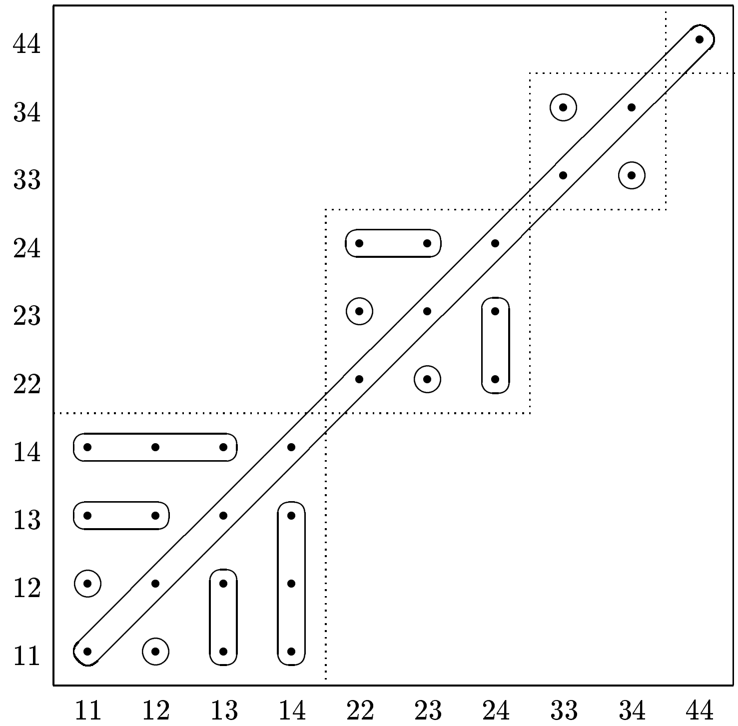

FIGURE 3. The partition associated to Ξ.

Next we prove (2) and (3) simultaneously. Suppose that $(a, b) \equiv_\Xi (c, d)$. Clearly if these pairs are equal, or if they are both in the diagonal, then we are done. Now suppose that case (iii) applies to (a, b) and (c, d). Thus $a_1 = b_1 = c_1 = d_1$, and $b_2, d_2 < a_2 = c_2$. If $a_1 = 1$, then both f and g map a, b, c, d to 11, and we are done. If not, then the elements b and d are mapped by both f and g into the set $H = \{(a_1 - 1, x) \mid a_1 - 1 \leq x \leq a_2 - 1\}$. Furthermore, $a_1 = b_1 \leq b_2 < a_2$ implies that $f(a) = g(a) = (a_1 - 1, a_2)$. Therefore the elements $(f(a), h)$ are in the same Ξ-class, when h runs over H, and so we are done. Finally if case (iv) applies to (a, b) and (c, d), then we can repeat the argument for case (iii) by switching the coordinates of the elements of T. Thus (2) and (3) are established as well.

Now to prove that Ξ is indeed a congruence suppose that $(a, b) \equiv_\Xi (c, d)$. We have to prove for every $(u, v) \in T$ that $(a, b) * (u, v) \equiv_\Xi (c, d) * (u, v)$ and $(u, v) * (a, b) \equiv_\Xi (u, v) * (c, d)$. We have the following cases.

If $u, v \in S = R - \{rr\}$, then $u * x = v * x = f(x)$ for every x, and so $(u, v) * (a, b) \equiv_\Xi (u, v) * (c, d)$ holds by (2). If either $u \notin S$ or $v \notin S$, then since $(u, v) \in T$ we have both $u, v \notin S$, so $u = v = rr$. In this case, $u * x = v * x =$

$g(x)$ for every x, and so (3) applies. Thus $(u,v)*(a,b) \equiv_\Xi (u,v)*(c,d)$ is proved in any case.

To prove that $(a,b)*(u,v) \equiv_\Xi (c,d)*(u,v)$ we first consider the case when $a,b,c,d \in S$. Then $(a,b)*(u,v) = (f(u),f(v)) = (c,d)*(u,v)$, and we are done. Since the case of $a=b=c=d=rr$ is trivial, we may assume without loss of generality that $a=b=rr$ and $c,d \in S$. Then $(a,b)*(u,v) = (g(u),g(v))$ and $(c,d)*(u,v) = (f(u),f(v))$. Thus (1) implies that $(a,b)*(u,v) \equiv_\Xi (c,d)*(u,v)$ indeed. We have therefore proved the claim, and so Ξ is indeed a congruence.

The congruence Ξ satisfies that the diagonal of $R \times R$ is a congruence block. This implies that $[1,\psi_{r-1}] = 0$ by a well-known argument (similar to the one used in the previous section). $\square$

Lemma 3.4. *The right nilpotence degree of* $\mathbf{R}$ *is exactly* r.

Proof. First we prove that the right nilpotence degree of $\mathbf{R}$ is at least r. We show by induction that $((i,i),(i,i+1)) \in [1_R)^{r-i+1}$ for $1 \le i < r$. Choose $t(x,y) = x*y$. First observe that

$$(1,1)*\underline{(1,1)} = f((1,1)) = (1,1) = g((1,1)) = (r,r)*\underline{(1,1)}\,.$$

Changing the underlined argument from $(1,1)$ to (r,r) we get that $(1,1)*(r,r) = f((r,r)) = (r-1,r)$ is related by $[1_R,1_R]$ to $(r,r)*(r,r) = g((r,r)) = (r-1,r-1)$. Thus our statement is established for $i = r-1$.

Now suppose that $((i,i),(i,i+1)) \in [1_R)^{r-i+1}$ for some $i > 1$. Then

$$(1,1)*\underline{(i,i+1)} = f((i,i+1)) = g((i,i+1)) = (r,r)*\underline{(i,i+1)}\,.$$

Changing the underlined argument to (i,i) we get that $(1,1)*(i,i) = f((i,i)) = (i-1,i)$ is related by $[[1_R)^{r-i+1},1]$ to $(r,r)*(i,i) = g((i,i)) = (i-1,i-1)$. Thus our statement is established for $i-1$.

The induction is complete, and so we have, for $i=1$, that the distinct elements $(1,1)$ and $(1,2)$ are related by $[1_R)^r$. Therefore this is not the zero congruence, and so the right nilpotence degree of $\mathbf{R}$ is at least r.

To finish the proof we must show that the right nilpotence degree of $\mathbf{R}$ is at most r. This follows if we prove that for the congruences ψ_i defined before Lemma 3.1 we have that $[\psi_i,1] \le \psi_{i-1}$ for $1 \le i \le r$.

Let C be a nontrivial class of ψ_i for some $1 \le i < r$. Then both f and g collapse C into a class of ψ_{i-1}. Since C is nontrivial, $C \subseteq S = R - \{rr\}$, and therefore the columns of the operation table of $*$ collapse C into a class of ψ_{i-1} as well. It is easy to deduce by direct calculation, or by invoking Lemma 4.3 of [6] that every polynomial p of $\mathbf{R}$ satisfies that $p(C_1,\dots,C_n)$ is contained in a ψ_{i-1}-block for every set $C_1,\dots,C_n$ of ψ_i-blocks. This already implies that $[\psi_i,1] \le \psi_{i-1}$. $\square$

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