MAL'TSEV CONDITIONS AND CENTRALITY

Excerpts from the book

The Shape of Congruence Lattices

Keith A. Kearnes, Emil W. Kiss

(in preparation).

META-THEOREM:

If a statement can be proved for locally finite varieties using tame congruence theory, then a possibly weaker form can be proved for arbitrary varieties, via calculations with Mal'tsev conditions.

EXAMPLE:

Theorem (D. Hobby, R. McKenzie)**.** *If a locally finite variety omits types* **1** *and* **5** *of tame congruence theory, and is residually small, then it is congruence modular.*

omits types **1** and **5** $\Longleftarrow$ satisfies a congruence identity

Theorem (K. Kearnes)**.** *If an arbitrary variety satisfies a nontrivial congruence identity, and is residually small, then it is congruence modular.*

Five aspects of the classification of varieties

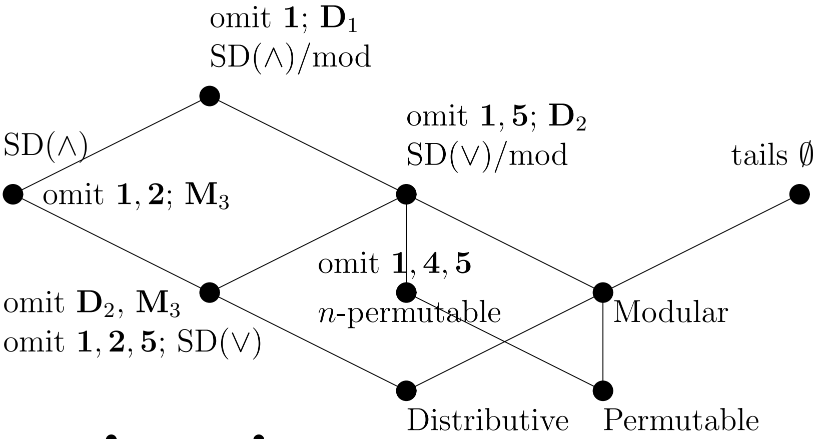
(1) **congruence-identities or implications:** distributivity, permutability, modularity, meet- and join-semidistributivity.

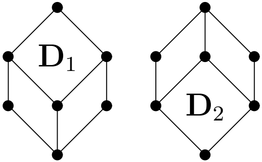
(2) **Mal'tsev-conditions:** the „strength” is measured by concrete varieties that should not satisfy the given condition (sets, modules, semilattices).

(3) **Omitting sublattices:** concrete lattices ($\mathbf{M}_3$, $\mathbf{N}_5$, $\mathbf{D}_1$, $\mathbf{D}_2$) that cannot occur as sublattices of congruence lattices of algebras in the variety.

(4) **Omitting types:** types of tame congruence theory that cannot occur as labels on congruence lattices of finite algebras in the variety.

(5) **Omitting special kinds of congruences or tolerances:** abelian, strongly abelian, orderable, „rectangular”, strongly rectangular.





The hierarchy of locfinite varieties

D. Hobby, R. McKenzie

Theorem. *For any locfinite variety $\mathcal{V}$, t.f.a. equivalent:*

(1) *The congruence lattice of every algebra in $\mathcal{V}$ has a congruence whose blocks are modular, and the factor-lattice is meet-semidistributive.*

(2) *$\mathcal{V}$ satisfies an idempotent Mal'tsev condition that fails in the variety of sets.*

(3) $\mathbf{D}_1 \notin \mathsf{S}\big(\mathrm{Con}(\mathbf{A})\big)$; $\mathbf{A} \in \mathcal{V}$.

(4) *$\mathcal{V}$ omits type* $\mathbf{1}$ *of tame congruence theory.*

(5) *No algebra in $\mathcal{V}$ can have a nontrivial strongly abelian congruence.*

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(6) *There is a ternary term of $\mathcal{V}$ that is Mal'tsev on the blocks of locally solvable congruences.*

(7) *Any two locally solvable congruences on algebras of $\mathcal{V}$ permute.*

Theorem. *For any locfinite variety $\mathcal{V}$, t.f.a. equivalent:*

(1) *The congruence lattice of every algebra in $\mathcal{V}$ is meet-semidistributive.*

(2) *$\mathcal{V}$ satisfies an idempotent linear Mal'tsev condition that fails in all nontrivial varieties of modules.*

(3) $\mathbf{M}_3 \notin \mathsf{S}\big(\mathrm{Con}(\mathbf{A})\big)$; $\mathbf{A} \in \mathcal{V}$.

(4) *$\mathcal{V}$ omits types $\mathbf{1}, \mathbf{2}$ of tame congruence theory.*

(5) *No algebra in $\mathcal{V}$ can have a nontrivial abelian congruence.*

Theorem. *For any locfinite variety $\mathcal{V}$, t.f.a. equivalent:*

(1) *The congruence lattice of every (finite) algebra in $\mathcal{V}$ has a congruence whose blocks are modular, and the factor-lattice is join-semidistributive.*

(2) *$\mathcal{V}$ satisfies an idempotent Mal'tsev condition that fails in the variety of semilattices.*

(3) $\mathbf{D}_2 \notin \mathsf{S}\left(\mathrm{Con}(\mathbf{A})\right)$; $\mathbf{A} \in \mathcal{V}$ ($\mathbf{A}$ *finite*).

(4) *$\mathcal{V}$ omits types $\mathbf{1}, \mathbf{5}$ of tame congruence theory.*

(5) *No algebra in $\mathcal{V}$ can have a nontrivial* **??????** *congruence.*

Theorem. *For any locfinite variety $\mathcal{V}$, t.f.a. equivalent:*

(1) *The congruence lattice of every (finite) algebra in $\mathcal{V}$ is join-semidistributive.*

(2) *$\mathcal{V}$ satisfies an idempotent linear Mal'tsev condition that fails in semilattices and in all nontrivial varieties of modules.*

(2') *There exist terms $d_0, \dots, d_{2n}$ such that in $\mathcal{V}$*

$$d_0(x,y,z) \approx x\,,\ d_{2n}(x,y,z) \approx z\,,$$

$$d_{2i-1}(x,y,x) \approx d_{2i}(x,y,x)\,,\ d_{2i-1}(x,y,y) \approx d_{2i}(x,y,y)\,,$$

$$d_{2i-2}(x,x,y) \approx d_{2i-1}(x,x,y) \quad (1 \le i \le n)\,.$$

(3) $\mathbf{D}_2, \mathbf{M}_3 \notin \mathsf{S}\big(\mathrm{Con}(\mathbf{A})\big)$; $\mathbf{A} \in \mathcal{V}$.

(4) *$\mathcal{V}$ omits types **1**, **2**, **5** of tame congruence theory.*

(5) ??????

PROBLEMS:

- What is the relationship between the listed properties and the existence of a congruence identity?
- Is it possible to remove the finiteness assumptions related to congruence join-semidistributivity?

Answered by *Keith A. Kearnes*.

- What is condition **??????** characterizing congruences that cannot occur in varieties omitting $\mathbf{D}_2$?

Answer: **rectangulation** with respect to a partial ordering. When the ordering is discrete, we get a concept related to strong abelianness (**strong rectangulation**).

- To what extent can this classification be generalized to the non-locally finite case?

Answer: to a great extent!

What is similar?

(1) Properties of congruence lattices like SD($\wedge$).

(2) Statements about the existence of Mal'tsev conditions failing in modules, semilattices, etc.

(2′) Concrete Mal'tsev conditions.

(3) Omitted sublattices ($\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{M}_3$).

(5) Characterizations involving omitted special congruences (strongly abelian, abelian, rectangular).

Theorem. *For an arbitrary variety $\mathcal{V}$, t.f.a. equivalent:*

(1) *$\mathcal{V}$ is congruence meet-semidistributive.*

(2) *$\mathcal{V}$ satisfies an idempotent Mal'tsev condition that fails in all nontrivial varieties of modules.*

(3) $\mathbf{M}_3 \notin \mathsf{S}\left(\mathrm{Con}(\mathbf{A})\right)$; $\mathbf{A} \in \mathcal{V}$.

(5) *No algebra in $\mathcal{V}$ can have a nontrivial abelian congruence/tolerance.*

(5') *The commutator of any two congruences is their intersection.*

(5'') $\mathbf{C}(\alpha, \beta; \delta)$ *holds if and only if*

$$\beta \wedge \left(\alpha \vee (\beta \wedge \delta)\right) \leq \delta\,.$$

Theorem. *For an arbitrary variety $\mathcal{V}$, t.f.a. equivalent:*

(1) *The congruence lattice of every algebra in $\mathcal{V}$ has a congruence whose blocks are modular, and the factor-lattice is join-semidistributive.*

(1') *$\mathcal{V}$ satisfies a nontrivial congruence identity.*

(2) *$\mathcal{V}$ satisfies an idempotent Mal'tsev condition that fails in the variety of semilattices.*

(3) $\mathbf{D}_2 \notin \mathsf{S}\left(\mathrm{Con}(\mathbf{A})\right)$; $\mathbf{A} \in \mathcal{V}$.

(5) *No algebra in $\mathcal{V}$ can have a nontrivial rectangular congruence/tolerance.*

Very complicated proof, based on the famous results by *Keith A. Kearnes* concerning the locally finite case.

Theorem. *For an arbitrary variety $\mathcal{V}$, t.f.a. equivalent:*

(1) *$\mathcal{V}$ is congruence join-semidistributive.*

(1') *$\mathcal{V}$ is congruence meet-semidistributive, and satisfies a nontrivial congruence identity.*

(2) *$\mathcal{V}$ satisfies an idempotent Mal'tsev condition that fails in semilattices and in all nontrivial varieties of modules.*

(2') *There exist terms $d_0, \ldots, d_{2n}$ such that in $\mathcal{V}$*

$$d_0(x,y,z) \approx x\,,\ d_{2n}(x,y,z) \approx z\,,$$

$$d_{2i-1}(x,y,x) \approx d_{2i}(x,y,x)\,,\ d_{2i-1}(x,y,y) \approx d_{2i}(x,y,y)\,,$$

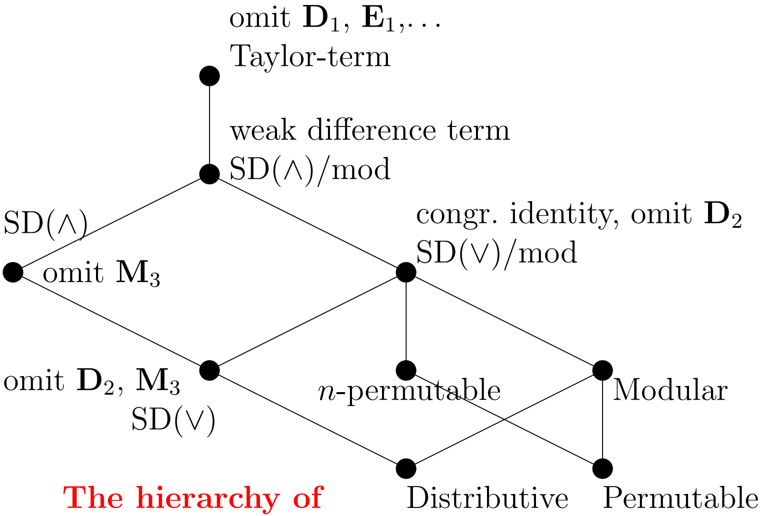
$$d_{2i-2}(x,x,y) \approx d_{2i-1}(x,x,y) \quad (1 \leq i \leq n)\,.$$

(3) $\mathbf{D}_2, \mathbf{M}_3 \notin \mathsf{S}\big(\operatorname{Con}(\mathbf{A})\big)$; $\mathbf{A} \in \mathcal{V}$.

(5) *No algebra in $\mathcal{V}$ can have a nontrivial rectangular or abelian congruence/tolerance.*

What is different?

- The methods of the proofs.
- No known generalization of omitting TCT-types.
- Only ad hoc methods to exclude individual sublattices of congruence lattices, or to guarantee centrality.
- the „peak" class (omitting $\mathbf{1}$; $\mathbf{D}_1$) splits: **weak difference term**.



The hierarchy of all varieties

A **Taylor-term** $F(x_1, \dots, x_n)$ is an idempotent term satisfying, for every $1 \le i \le n$, an identity of the form

$$F(x_1, \dots, x_n) = F(y_1, \dots, y_n)\,,$$

where all $x_j, y_j \in \{x, y\}$, and $\{x_i, y_i\} = \{x, y\}$ (so $x_i \neq y_i$).

Exists in every variety satisfying an idempotent Mal'tsev condition that fails in the variety of sets. (The weakest possible idempotent Mal'tsev condition.)

A **weak difference term** $d(x, y, z)$ on $\mathbf{A}$ is defined to satisfy $d(a, a, b)\ [\alpha, \alpha]\ b$ and $d(a, b, b)\ [\alpha, \alpha]\ a$ for every $(a, b) \in \alpha \in \mathrm{Con}(\mathbf{A})$.

Exists in every locally finite variety with a Taylor-term (that is, which omits $\mathbf{1}$).

Exists in every variety satisfying an idempotent Mal'tsev condition that fails in semilattices.

Bad example:

Let $\mathbf{A}$ be the real interval $[0, 1] \subseteq \mathbb{R}$, equipped with the basic operation $F(x, y) = (x + y)/2$.

$F(x, y) = F(y, x)$ is a Taylor-identity. Since F is an affine operation, $\mathbf{A}$ is abelian.

Let α have blocks $[0, 1)$ and $\{1\}$. This is a congruence, and $\mathbf{A}/\alpha$ is the two-element semilattice, which is not even solvable.

The variety generated by $\mathbf{A}$ has no weak difference term. Indeed, $\mathbf{A}$ is Abelian, hence a weak difference term would be a Mal'tsev term. But the two-element semilattice has no Mal'tsev term.

Earlier results by *Keith A. Kearnes, Ágnes Szendrei.*
Main tool: symmetric = linear commutator.

Theorem. *Let $\mathcal{V}$ be a variety with a Taylor-term.*

(1) *The abelian algebras in $\mathcal{V}$ are quasi-affine (subalgebras of reducts of modules).*
(2) *$\mathcal{V}$ satisfies a congruence equation in $\wedge, \vee, \circ$.*
(3) *If $\mathcal{V}$ has no abelian congruences, then it is congruence meet-semidistributive.*

Let $\mathcal{V}$ be a variety with a weak difference term.

(4) *The abelian algebras in $\mathcal{V}$ are affine (polynomially equivalent to a module).*

The existence of a weak difference term is the weakest possible idempotent Mal'tsev condition implying that abelian algebras are affine.

Theorem. *For an arbitrary variety $\mathcal{V}$, t.f.a. equivalent:*

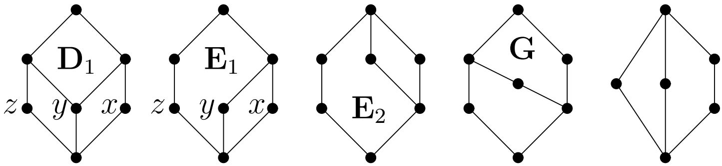
(1) $\mathcal{V}$ *satisfies the following congruence implication:*
$(x \wedge y) = (x \vee y) \wedge z \Longrightarrow (x \wedge y) = (x \vee z) \wedge y$.

(2) $\mathcal{V}$ *satisfies an idempotent Mal'tsev condition that fails in the variety of sets (= has a Taylor-term).*

(3) $\mathbf{D}_1 \notin \mathsf{S}\big(\operatorname{Con}(\mathbf{A})\big);\ \mathbf{A} \in \mathcal{V}$.

(5) *No algebra in* $\mathcal{V}$ *can have a nontrivial strongly abelian congruence/tolerance.*

(5') *No algebra in* $\mathcal{V}$ *can have a nontrivial strongly rectangular congruence/tolerance.*



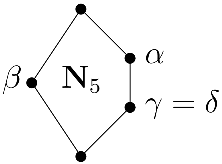
All excluded

Theorem. *Suppose that $\mathcal{V}$ has a Taylor-term, $\mathbf{A} \in \mathcal{V}$.*

(1) *If $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$, then $\mathbf{C}(\alpha, \alpha; \alpha \wedge \beta)$ is equivalent to $\mathbf{C}(\alpha \vee \beta, \alpha \vee \beta; \beta)$.*

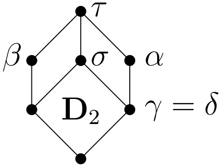
(2) *If $\alpha, \beta, \gamma, \delta \in \mathrm{Con}(\mathbf{A})$ such that $\mathbf{C}(\beta \vee \gamma, \alpha; \delta)$ and $\alpha \cap (\beta \circ \gamma) \cap (\gamma \circ \beta) \subseteq \delta \subseteq \alpha \subseteq \beta \vee \gamma$, then $\delta = \alpha$.*

(3) *The following lattices with the given centrality relations cannot occur as sublattices of $\mathrm{Con}(\mathbf{A})$:*



$\mathbf{C}(\beta, \alpha; \delta)$ or

$\mathbf{C}(\beta, \beta; \beta \wedge \delta)$



$\mathbf{C}(\tau, \tau; \sigma)$

Sketch of proof of (2). Needed: if $\mathbf{C}(\beta \vee \gamma, \alpha; \delta)$ and $\alpha \cap (\beta \circ \gamma) \cap (\gamma \circ \beta) \subseteq \delta \subseteq \alpha \subseteq \beta \vee \gamma$, then $\delta = \alpha$.

Suppose $(u, v) \in \alpha - \delta$. Then there is a chain like

$$u = u_0 \; \beta \; u_1 \; \gamma \; u_2 \; \beta \; u_3 \; \gamma \; u_4 \; \beta \; u_5 = v\,.$$

Let $F(x, y) = F(y, x)$ be a Taylor-term, then

$$F(u_1, u_0) \; \gamma \; F(u_2, u_0) \; \beta \; F(u_3, u_1) \; \gamma \; F(u_4, u_1) \; \beta \; F(u_5, u_1)$$

is a shorter chain between $F(u_1, u)$ and $F(v, u_1)$. But

$$r := F(u_1, u) = F(u, u_1) \; \alpha \; F(v, u_1) =: s\,.$$

Thus $(r, s) \in \alpha$ can be connected by a shorter chain. By induction we may assume that $(r, s) \in \delta$. (If the original chain is of length 2, then we use the assumption $\alpha \cap (\beta \circ \gamma) \cap (\gamma \circ \beta) \subseteq \delta$ instead of the trick above).

Now use the centrality $\mathbf{C}(\beta \vee \gamma, \alpha; \delta)$: from

$$r = F(u, \underline{u_1}) \ \delta \ F(v, \underline{u_1}) = s$$

we obtain that

$$u = F(u, \underline{u}) \ \delta \ F(v, \underline{u}) \, .$$

Switching the variables of F the same argument implies that $F(v, u) \ \delta \ v$, so $u \ \delta \ v$ and we are done.

A congruence α of $\mathbf{A}$ is **∞-solvable**, if there exists an ordinal κ, and congruences α_i $(i \leq \kappa)$ such that

(1) $\alpha_0 = 0_A$ and $\alpha_\kappa = \alpha$;
(2) α_{i+1}/α_i is abelian for every i;
(3) if j is a limit ordinal, then $\alpha_j = \bigcup\{\alpha_i \mid i < j\}$.

If $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$, then they are **solvably related** (in notation $\alpha \overset{\mathrm{s}}{\sim} \beta$), if $(\alpha \vee \beta)/(\alpha \wedge \beta)$ is an ∞-solvable congruence of $\mathbf{A}/(\alpha \wedge \beta)$.

It is important to have a well-ordered chain: a noncommutative free group is a subdirect product of finite solvable groups (the commutator series intersects to zero), but it is not ∞-solvable.

The variety of rings defined by $x^2 \approx 0$ has the property that each member is ∞-solvable.

Theorem. *Suppose that $\mathcal{V}$ has a weak difference term.*

(1) *The relation* $\overset{s}{\sim}$ *is a congruence on the congruence lattice of every* $\mathbf{A} \in \mathcal{V}$*, which is compatible with the complete join operation. Each block of this congruence is modular, and* $\mathrm{Con}(\mathbf{A})/\overset{s}{\sim}$ *is a complete meet-semidistributive lattice.*

(3) *If* $\mathbf{N}_5 \leq \mathrm{Con}(\mathbf{A})$, $\mathbf{A} \in \mathcal{V}$*, then its bottom element is not solvably related to* β *(labeling as above).*

(3') *If* $\mathbf{D}_2 \leq \mathrm{Con}(\mathbf{A})$, $\mathbf{A} \in \mathcal{V}$*, then no two elements of* $\mathbf{D}_2$ *can be solvably related.*

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(6) *Homomorphic images of abelian (∞-solvable) algebras and congruences are abelian (∞-solvable).*

(7) *Any two solvably related congruences permute.*

(8) *Every ∞-solvable reflexive compatible binary relation is a congruence.*

Photos by Piotr Zgodzinski (http://www.jubi.buum.pl) and Ádám Pethő.