Abelian Algebras and the Hamiltonian Property *

Emil W. Kiss   Matthew A. Valeriote †

Abstract

In this paper we show that a finite algebra $\mathbf{A}$ is Hamiltonian if the class $\mathsf{HS}(\mathbf{A}^A)$ consists of Abelian algebras. As a consequence, we conclude that a locally finite variety is Abelian if and only if it is Hamiltonian. Furthermore it is proved that $\mathbf{A}$ generates an Abelian variety if and only if $\mathbf{A}^{A^3}$ is Hamiltonian. An algebra is Hamiltonian if every nonempty subuniverse is a block of some congruence on the algebra and an algebra is Abelian if for every term $t(x, \bar{y})$, the implication $t(x, \bar{y}) = t(x, \bar{z}) \to t(w, \bar{y}) = t(w, \bar{z})$ holds. Thus, locally finite Abelian varieties have definable principal congruences, enjoy the congruence extension property, and satisfy the RS-conjecture.

1 Introduction

Abelian algebras have played an important role in the development of universal algebra over the last decade. An algebra is said to be Abelian if every one of its terms satisfies a particular universally quantified implication which will be stated shortly; in the past this condition has been called the *term condition*, or TC for short.

TC was first used in universal algebra by Werner, Lampe and McKenzie. Werner [23] used TC in his investigations of skew congruences and functionally complete algebras. His results were improved by McKenzie [14], where it is shown that in congruence permutable varieties, finite simple algebras are functionally complete if they fail to satisfy TC. In the same paper it is

*1980 *Mathematical Subject Classification* (1985 *Revision*), Primary 08A05; Secondary 03C13.

†Support of NSERC is gratefully acknowledged

proved that a minimal locally finite congruence permutable variety must be congruence distributive or all of the algebras in it must be Abelian. Later, C. Herrmann proved this for congruence modular varieties (see [7]).

TC showed up in Lampe's proof ([20]) that some algebraic lattices could not be represented as the congruence lattice of an algebra having only a few fundamental operations. A key lemma is that if $\mathbf{L} = \text{Con}\,\mathbf{A}$ and for all compact ψ in $\mathbf{L}$ there are congruences θ and ϕ satisfying

$$\theta \vee \phi \geq \psi \quad \text{and} \quad \theta \wedge \psi = \phi \wedge \psi = 0_A\,,$$

then $\mathbf{A}$ must satisfy TC.

The reader not familiar with the basic definitions and notions from universal algebra may wish to consult [5] or [18].

Definition 1.1 Let $\mathbf{A}$ be an algebra. We say that $\mathbf{A}$ is Abelian (or satisfies TC) if for all terms $t(x, \bar{y})$ of $\mathbf{A}$ and all elements a, b, $\bar{c}$ and $\bar{d}$ in A we have

$$t^{\mathbf{A}}(a, \bar{c}) = t^{\mathbf{A}}(a, \bar{d}) \longrightarrow t^{\mathbf{A}}(b, \bar{c}) = t^{\mathbf{A}}(b, \bar{d}).$$

A class of algebras is said to be Abelian if every algebra in it is Abelian.

This definition can be seen to be a generalization of what it means for a group to be Abelian. It is not hard to show that a group is Abelian in the above sense iff its multiplication is commutative. Just as easy to prove is that any module or essentially unary algebra is also Abelian.

In a congruence modular variety, being Abelian has strong structural consequences. If $\mathbf{A}$ is Abelian and generates a congruence modular variety, then $\mathbf{A}$ is polynomially equivalent to a left $\mathbf{R}$-module for some ring $\mathbf{R}$ with identity. We call such an algebra *affine*. This was proved by C. Herrmann [8] and arose out of his study of the commutator in congruence modular varieties. The term condition and its relativization to congruences has played a major role in the development of the commutator, in fact the commutator can be defined in terms of a relativized term condition [7]. The detailed description of congruence modular Abelian varieties given by commutator theory has been used by several authors, most notably by Burris and McKenzie [4] in their study of decidability and by Baldwin and McKenzie [2] in their study of varieties with small spectrum.

The next milestone in the development of the Abelian property was McKenzie's paper [15]. He proved that finite algebras having a certain type of congruence lattice must satisfy a condition he called TC*, which is a stronger form of the Abelian property. He was able to use this property to exhibit an infinite class of finite lattices that could not be represented as the congruence lattice of a finite algebra having only one basic operation.

Definition 1.2 Let $\mathbf{A}$ be an algebra. We say that $\mathbf{A}$ is strongly Abelian (or satisfies TC*) if for all terms $t(x, \bar{y})$ of $\mathbf{A}$ and all elements a, b, e, $\bar{c}$ and $\bar{d}$ in A we have

$$t^{\mathbf{A}}(a, \bar{c}) = t^{\mathbf{A}}(b, \bar{d}) \longrightarrow t^{\mathbf{A}}(e, \bar{c}) = t^{\mathbf{A}}(e, \bar{d}).$$

A class of algebras is said to be strongly Abelian if every algebra in it is strongly Abelian.

The basic examples for this condition are unary algebras and their so called matrix powers. These results of McKenzie were the starting point of the development of tame congruence theory, which is a powerful tool for investigating finite algebras. It was developed in the 80's by McKenzie and his student D. Hobby [9]. Both the Abelian and strongly Abelian properties play a major role in this theory.

In general, Abelian algebras need not behave nicely at all. This is best seen by observing that given a type τ of algebras and a set X of variables, the term algebra of type τ generated by X is also Abelian. This example points out that the property of being Abelian is not preserved in general under homomorphic images. Since this property is defined by a set of universal Horn formulas, it is preserved under subalgebras and direct products. For a general discussion of Abelian algebras the reader is encouraged to look at [3] and [9].

In this paper we are primarily interested in finding the structural consequences for a finite algebra when it is assumed to generate an Abelian variety. Even under this moderately strong assumption, the variety can still be widely misbehaved, for instance the second author has found a finite algebra $\mathbf{A}$ such that $\mathbf{A}$ generates a residually large (see section 4 of this paper), undecidable Abelian variety (see [11]).

A question posed in [9] and first found in [3] asks if a finite algebra generating an Abelian variety must be Hamiltonian. The Hamiltonian property was introduced by Csákány [6] and by Shoda [21], and is a natural generalization of the concept of a Hamiltonian group.

Definition 1.3 An algebra $\mathbf{A}$ is called Hamiltonian if every nonempty subuniverse of $\mathbf{A}$ is a block of some congruence on $\mathbf{A}$. A variety is called Hamiltonian if every one of its members is.

This property is related to the Abelian property as is shown in the following, well–known theorem. For its easy proof one has to apply the Hamiltonian property to the diagonal subuniverse of $\mathbf{A}^2$.

THEOREM 1.4 *Let* $\mathbf{A}$ *be an algebra such that* $\mathbf{A}^2$ *is Hamiltonian. Then* $\mathbf{A}$ *is Abelian.*

COROLLARY 1.5 *Every Hamiltonian variety is Abelian.*

One can give a translation of the Hamiltonian property using the term functions of the algebra. The reader can easily verify that an algebra $\mathbf{A}$ is Hamiltonian iff for each term $t(x, \bar{y})$ of $\mathbf{A}$ and elements a, b, $\bar{c}$ in A there exists a ternary term r of $\mathbf{A}$ such that $r(a, b, t(a, \bar{c})) = t(b, \bar{c})$. To have $\mathbf{A}$ generate a Hamiltonian variety we must have a uniform r, depending only on t, but not on the elements a, b, $\bar{c}$. This is a result of Klukovits [13].

THEOREM 1.6 *A variety* $\mathcal{V}$ *is Hamiltonian if and only if for every term* $t(x, \bar{z})$ *there is a 3-ary term* $r_t(x, y, u)$ *such that*

$$\mathcal{V} \models r_t(x, y, t(x, \bar{z})) \approx t(y, \bar{z}).$$

Such a term r_t *will be called a Klukovits term for* t.

We have seen that Hamiltonian varieties are Abelian, it is easy to prove this by using Klukovits's characterization. If the variety is strongly Abelian, then it can be characterized by the following stronger form of the Klukovits property.

THEOREM 1.7 *A variety* $\mathcal{V}$ *is Hamiltonian and strongly Abelian if and only if for every term* $t(x, \bar{z})$ *there is a binary term* $r_t(y, u)$ *such that*

$$\mathcal{V} \models r_t(y, t(x, \bar{z})) \approx t(y, \bar{z})$$

(that is, a Klukovits term not depending on its first variable).

As we have seen, the local form of the Klukovits property yields the Hamiltonian property for single algebras. We shall need the 'strong' version of this condition.

Definition 1.8 An algebra $\mathbf{A}$ is called strongly Hamiltonian if for each term $t(x, \bar{y})$ of $\mathbf{A}$ and elements a, b, $\bar{c}$ in A there exists a binary term r of $\mathbf{A}$ such that $r(b, t(a, \bar{c})) = t(b, \bar{c})$.

It is not hard to see that if $\mathbf{A}^2$ is strongly Hamiltonian, then $\mathbf{A}$ is strongly Abelian, and a variety $\mathcal{V}$ is strongly Abelian and Hamiltonian iff every member of $\mathcal{V}$ is strongly Hamiltonian. The strong Hamiltonian property is clearly preserved by subalgebras and homomorphic images.

In attempting to answer the above mentioned problem of deciding if locally finite Abelian varieties are Hamiltonian, several notable partial results have been obtained. The authors in [12] have shown that if a finite algebra generates a strongly Abelian variety, then it is Hamiltonian. Valeriote [22] has shown that if $\mathbf{A}$ is a finite simple Abelian algebra, then it is Hamiltonian. McKenzie [17] generalized this result by proving that if B is a maximal proper subuniverse of a finite algebra $\mathbf{A}$ such that every quotient of $\mathbf{A}$ is Abelian, then B is a block of some congruence of $\mathbf{A}$. Both of these latter two results use tame congruence theory.

We shall prove in Corollary 2.5 that if $\mathbf{A}$ is a finite algebra and $\mathsf{HS}(\mathbf{A}^A)$ is Abelian, then $\mathbf{A}$ is Hamiltonian. Our proof is elementary. This settles the original problem, but our result does not generalize the ones mentioned above that use tame congruence theory. However, at the end of the paper we shall present an example of a five element algebra showing that from the assumption that $\mathsf{HS}(\mathbf{A})$ is Abelian it does not follow that $\mathbf{A}$ is Hamiltonian. Thus, McKenzie's theorem cannot be generalized in the obvious way.

In Section 3 of this paper we will provide an effective characterization of those finite algebras $\mathbf{A}$ that generate an Abelian variety. The characterization is effective since it involves only checking whether certain finite powers of $\mathbf{A}$ are Hamiltonian. Unfortunately, the algorithm as presented is extremely inefficient since it relies on checking whether the algebra $\mathbf{A}^{A^3}$ is Hamiltonian or not. In that section we also provide examples that show that in general we can't get away with just considering small powers of the algebra $\mathbf{A}$, not even in the strongly Abelian case.

2 Hamiltonian Algebras

In this section we will find a condition on an algebra that ensures that it is Hamiltonian. Before we begin, we would like to state a useful and easy to prove property of Abelian algebras. By the kernel of an n-ary function f defined on a set A we mean the partition of A^n induced by f in the usual way.

LEMMA 2.1 *Let $\mathbf{A}$ be an Abelian algebra and let $p(x_1, \ldots, x_n, y)$ be a polynomial of $\mathbf{A}$. Then for all a, b from A, the operations $p(x_1, \ldots, x_n, a)$ and $p(x_1, \ldots, x_n, b)$ have the same kernel. As a result, if S is a finite subset of A, then $p(S^n, a)$ and $p(S^n, b)$ have the same size.*

For $\mathbf{A}$ an algebra and $S \subseteq A$, let $\mathcal{P}(S)$ be the poset of all subsets of A of the form $p(S, S, \ldots, S)$, where p is some polynomial of $\mathbf{A}$, ordered by inclusion. We will show that if for some N, all of the elements of $\mathcal{P}(S)$ have size at most N and if the class $\mathsf{HS}(\mathbf{A}^N)$ is Abelian, then the maximal elements of $\mathcal{P}(S)$ have special properties.

For the next two lemmas, let us assume that N is a natural number, $\mathbf{A}$ is an algebra and S is a subset of A such that $\mathsf{HS}(\mathbf{A}^N)$ is Abelian and all of the elements of $\mathcal{P}(S)$ have at most N elements. Let β be the congruence of $\mathbf{A}$ generated by identifying all of the elements of S. Note that every element of $\mathcal{P}(S)$ is contained in some β-class.

LEMMA 2.2 *Let $T \in \mathcal{P}(S)$ be a maximal element, and suppose that $T = t(S, S, \ldots, S, a_1, \ldots, a_n)$ for some polynomial t of $\mathbf{A}$ and some elements a_i from A. If c_i, $1 \leq i \leq n$, is a string of elements from A such that $\langle a_i, c_i \rangle \in \beta$ for all $1 \leq i \leq n$, then $T = t(S, S, \ldots, S, c_1, \ldots, c_n)$.*

PROOF. Clearly it suffices to prove this for the case $n = 1$, since t is assumed to be a polynomial of $\mathbf{A}$. Since a_1 and c_1 are β-related and β is generated by the set S, there exists, for some m, a sequence of polynomials $p_1(x)$, $\ldots$, $p_m(x)$ of $\mathbf{A}$ and elements s_i, s'_i from S, for $1 \leq i \leq m$, such that $p_1(s_1) = a_1$, $p_m(s'_m) = c_1$ and, for $i < m$, $p_i(s'_i) = p_{i+1}(s_{i+1})$.

We will show by induction that, for $1 \leq i \leq m$, $T = t(S, S, \ldots, S, p_i(s))$ for all $s \in S$. Since $p_m(s'_m) = c_1$, this will be enough to prove the lemma. Suppose that for some $1 \leq i \leq m$ we have that $T = t(S, S, \ldots, S, p_i(s_i))$ (by assumption this is true for $i = 1$). Since T is maximal in $\mathcal{P}(S)$, then we have that $T = t(S, S, \ldots, S, p_i(S))$. Using Lemma 2.1, the fact that $\mathbf{A}$ is Abelian implies that for all $s \in S$, in particular for $s = s_i'$, we have $T = t(S, S, \ldots, S, p_i(s))$.

Since we have that $p_{i+1}(s_{i+1}) = p_i(s_i')$, then we also have that

$$T = t(S, S, \ldots, S, p_{i+1}(s_{i+1})).$$

This allows us to conclude that $T = t(S, S, \ldots, S, p_{i+1}(s))$ for all $s \in S$. ∎

If r is a polynomial of an algebra $\mathbf{A}$, and $\mathbf{C}$ is a subalgebra of a direct power of $\mathbf{A}$, then $r^{\mathbf{C}}$ denotes the mapping defined on $\mathbf{C}$ by letting r act componentwise. It is easy to see that if C contains the diagonal, then $r^{\mathbf{C}}$ is a polynomial of $\mathbf{C}$.

LEMMA 2.3 *Let $T \in \mathcal{P}(S)$ be maximal and suppose that*

$$T = t(S, S, \ldots, S, a_1, \ldots, a_n)$$

for some polynomial $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ of $\mathbf{A}$ and elements a_i from A. Let c_i, $1 \leq i \leq n$, be elements of A such that the set $t(S, S, \ldots, S, c_1, \ldots, c_n)$ lies in the same β-class as T. Then

$$T = t(S, S, \ldots, S, c_1, \ldots, c_n).$$

Proof. By way of contradiction assume that the set

$$T_{\bar{c}} = t(S, S, \ldots, S, c_1, \ldots, c_n)$$

is different from T. Then there are some elements $s_1, \ldots, s_m$ of S such that

$$0_{\bar{a}} = t(\bar{s}, \bar{a}) \neq t(\bar{s}, \bar{c}) = 0_{\bar{c}}.$$

By assumption we know that $\langle 0_{\bar{a}}, 0_{\bar{c}} \rangle \in \beta$.

Assume that T has k elements and let $\vec{T} = \langle t_1, \ldots, t_k \rangle \in A^k$ be some listing of T. For each $1 \leq i \leq k$, choose some elements s_j^i, $1 \leq j \leq m$, from S such that $t_i = t(s_1^i, \ldots, s_m^i, \bar{a})$. Let $\vec{S}_j = \langle s_j^1, s_j^2, \ldots, s_j^k \rangle \in A^k$, for each $1 \leq j \leq m$.

Let $\mathbf{C}$ be the subalgebra of $\mathbf{A}^k$ generated by the sets $\{\vec{S}_i : 1 \leq i \leq m\}$ and $\{\hat{a} : a \in A\}$, where for $a \in A$, $\hat{a}$ is the k-tuple $\langle a, a, \ldots, a \rangle$. For $\bar{d}$ an

n-tuple from A, let $\vec{T}_{\bar{d}} = t^{\mathbf{C}}(\vec{S}_1, \ldots, \vec{S}_m, \hat{d}_1, \ldots, \hat{d}_n)$. By this notation, $\vec{T} = \vec{T}_{\bar{a}}$ and $\vec{T}_{\bar{c}}$ is some listing of the elements of $T_{\bar{c}}$.

In the algebra $\mathbf{C}$ we have

$$\begin{array}{rcl} t^{\mathbf{C}}(\hat{s}_1, \ldots, \hat{s}_m, \hat{a}_1, \ldots, \hat{a}_n) & = & \hat{0}_{\bar{a}} \\ t^{\mathbf{C}}(\hat{s}_1, \ldots, \hat{s}_m, \hat{c}_1, \ldots, \hat{c}_n) & = & \hat{0}_{\bar{c}} \\ & \text{and} & \\ t^{\mathbf{C}}(\vec{S}_1, \ldots, \vec{S}_m, \hat{a}_1, \ldots, \hat{a}_n) & = & \vec{T}_{\bar{a}} \\ t^{\mathbf{C}}(\vec{S}_1, \ldots, \vec{S}_m, \hat{c}_1, \ldots, \hat{c}_n) & = & \vec{T}_{\bar{c}}. \end{array}$$

By our assumptions on $\mathbf{A}$, all of the quotients of $\mathbf{C}$ should be Abelian, but we will now demonstrate that $\mathbf{C}$ modulo the congruence θ generated by the pair $\langle \hat{0}_{\bar{a}}, \hat{0}_{\bar{c}} \rangle$ is not Abelian. It is clear from the above equalities that to prove this, it will suffice to show that $\langle \vec{T}_{\bar{a}}, \vec{T}_{\bar{c}} \rangle$ does not belong to θ. The following claim will establish this.

Claim 1 *If $\vec{v} = \langle v_1, \ldots, v_k \rangle \in C$ is θ-related to $\vec{T}_{\bar{a}}$, then $\{v_1, \ldots, v_k\} = T$.*

From the characterization of principal congruences due to Mal'cev, it will be enough to show that for all polynomials $p(x)$ of $\mathbf{C}$, if one of $p(\hat{0}_{\bar{a}})$, $p(\hat{0}_{\bar{c}})$ is a one–to–one listing of the elements of T, then both of them are. Suppose, without loss of generality, that $p(\hat{0}_{\bar{a}})$ is of this form, for p a polynomial of $\mathbf{C}$. Then there is a polynomial $r(x, y_1, \ldots, y_m)$ of $\mathbf{A}$ such that $p(x) = r^{\mathbf{C}}(x, \vec{S}_1, \ldots, \vec{S}_m)$ for all x from C, where $r^{\mathbf{C}}$ is the polynomial of $\mathbf{C}$ obtained by applying r componentwise.

So we have that $p(\hat{0}_{\bar{a}}) = r^{\mathbf{C}}(\hat{0}_{\bar{a}}, \vec{S}_1, \ldots, \vec{S}_m)$ is a listing of T. Using the maximality of T we conclude that $r(0_{\bar{a}}, S, S, \ldots, S) = T$. Since $0_{\bar{c}}$ is β-related to $0_{\bar{a}}$, then by the previous lemma we have that $T = r(0_{\bar{c}}, S, S, \ldots, S)$. Since $\mathbf{A}$ is Abelian and S is finite, then this implies that in $\mathbf{C}$, the k-tuple $r^{\mathbf{C}}(\hat{0}_{\bar{c}}, \vec{S}_1, \ldots, \vec{S}_m)$ is also a listing of T.

This ends the proof of the claim and of the lemma. ∎

THEOREM 2.4 *Let N be a natural number and $\mathbf{A}$ an algebra such that $\mathsf{HS}(\mathbf{A}^N)$ is Abelian. Then every nonempty subuniverse of $\mathbf{A}$ having at most N elements is a block of some congruence on $\mathbf{A}$.*

PROOF. Let B be a nonempty subuniverse of $\mathbf{A}$ having at most N elements. Let β be the congruence of $\mathbf{A}$ generated by identifying all of the elements of B.

The first step of this proof will be to establish that all of the elements of the poset $\mathcal{P}(B)$ have at most N elements. Let $T = t(B, B, \ldots, B)$ for some polynomial t of $\mathbf{A}$. There is some term operation $r(\bar{x}, y_1, \ldots, y_m)$ of $\mathbf{A}$ and some elements $a_1, \ldots, a_m$ from A such that $t(\bar{x}) = r(\bar{x}, \bar{a})$ for all $\bar{x}$ from A. Let $\bar{b}$ be a string of m elements from B. Then since $\mathbf{A}$ is Abelian, it follows that the sets $r(B, B, \ldots, B, \bar{a})$ and $r(B, B, \ldots, B, \bar{b})$ have the same size. Since r is a term operation and B is a subuniverse of $\mathbf{A}$, we know that $r(B, B, \ldots, B, \bar{b})$ is a subset of B and so has at most N elements. Thus, the set T has at most N elements, too.

Now, if B is not a block of β, then there must be some unary polynomial $p(x)$ of $\mathbf{A}$ such that $p(B) \not\subseteq B$ and $p(B) \cap B \neq \emptyset$. Choose some maximal member, T, of $\mathcal{P}(B)$ that contains $p(B)$. So, T has nonempty intersection with B and is not contained in B. Let $t(\bar{x}, \bar{y})$ be a term operation of $\mathbf{A}$ and let $\bar{a}$ be elements from A such that $T = t(B, B, \ldots, B, \bar{a})$. If we substitute the string $\bar{a}$ by a string $\bar{b}$ of elements from B, then the resulting set $T' = t(B, B, \ldots, B, \bar{b})$ is contained within B. Since B^2 generates the congruence β and since $T \cap B \neq \emptyset$, then T and T' lie in the same β-class. But then from the previous lemma we conclude that $T = T'$. This is a contradiction, and so B must be a block of β. ∎

COROLLARY 2.5 *Let $\mathbf{A}$ be a finite algebra. If $\mathsf{HS}(\mathbf{A}^A)$ is Abelian, then $\mathbf{A}$ is Hamiltonian.*

COROLLARY 2.6 *Let $\mathcal{V}$ be a locally finite variety. Then $\mathcal{V}$ is Abelian if and only if it is Hamiltonian.*

PROOF. One direction of this is Corollary 1.5. The other direction follows from the previous corollary once one makes the observation that if a locally finite variety fails to be Hamiltonian, then some finite member of it is not Hamiltonian. ∎

3 Abelian Varieties

In this section we provide an algorithm to determine when a finite algebra generates an Abelian variety.

THEOREM 3.1 *Let* $\mathbf{A}$ *be a finite algebra of size* k*. Then* $\mathsf{V}(\mathbf{A})$ *is Abelian if and only if* $\mathbf{A}^{k^3}$ *is Hamiltonian. For every integer* $n \geq 4$ *there exists a finite algebra* $\mathbf{A}$ *of* $2n+2$ *elements such that* $\mathbf{A}^{n-2}$ *is Hamiltonian,* $\mathsf{HS}(\mathbf{A}^{[n/2]-1})$ *is strongly Abelian and* $\mathsf{V}(\mathbf{A})$ *is not Abelian.*

A consequence of this is that there is no bound N independent of the size of $\mathbf{A}$ such that $\mathsf{V}(\mathbf{A})$ is Abelian iff $\mathsf{HS}(\mathbf{A}^N)$ is Abelian. On the other hand, since there is an effective way to check whether a finite algebra is Hamiltonian this theorem gives (an unreasonably slow) algorithm to check if $\mathsf{V}(\mathbf{A})$ is Abelian.

PROOF. Let $\mathbf{A}$ be finite of k elements. By Corollary 2.6 if $\mathsf{V}(\mathbf{A})$ is Abelian, then it is Hamiltonian, thus so is $\mathbf{A}^{k^3}$.

Conversely, assume that $\mathbf{A}^{k^3}$ is Hamiltonian and let $f(x, z_1, \ldots, z_n)$ be a term of $\mathbf{A}$. According to Theorem 1.6, in order to show that $\mathbf{A}$ generates an Abelian variety it will be enough to find a Klukovits term for f. For $x \in A$ set $U_x = f^{\mathbf{A}}(x, A, \ldots, A)$, and for every $u \in U_x$ pick $z_i^{x,u}$ such that $f^{\mathbf{A}}(x, z_1^{x,u}, \ldots, z_n^{x,u}) = u$. Let J be the set of all triples $(x, y, u) \in A^3$ such that $u \in U_x$. Then $\mathbf{A}^J$ is isomorphic to a subalgebra of $\mathbf{A}^{k^3}$, so it is Hamiltonian.

If we define the elements $\bar{x}, \bar{y}, \bar{u}, \bar{z}_i$ of A^J so that for each $j = (x, y, u) \in J$, the j-th coordinates of these elements are $x, y, u, z_i^{x,u}$, respectively, then we have $f^{\mathbf{A}^J}(\bar{x}, \bar{z}_1, \ldots, \bar{z}_n) = \bar{u}$. Let $\bar{v} = f^{\mathbf{A}^J}(\bar{y}, \bar{z}_1, \ldots, \bar{z}_n)$ and consider the subalgebra of $\mathbf{A}^J$ generated by $\bar{x}, \bar{y}, \bar{u}$. By the Hamiltonian property, $\bar{v}$ is an element of this subalgebra (since $\bar{u}$ and $\bar{v}$ are congruent modulo the principal congruence generated by $\bar{x}$ and $\bar{y}$). Let r be a term satisfying $r^{\mathbf{A}^J}(\bar{x}, \bar{y}, \bar{u}) = \bar{v}$. We will show that r is a Klukovits term for f, that is, $\mathbf{A}$ satisfies the identity

$$r(x, y, f(x, z_1, \ldots, z_n)) \approx f(y, z_1, \ldots, z_n).$$

If we set $u = f^{\mathbf{A}}(x, z_1, \ldots, z_n)$ and $v = f^{\mathbf{A}}(y, z_1, \ldots, z_n)$, then $j = (x, y, u) \in J$. Let v' denote the j-th component of $\bar{v}$. By the definition of r we have that $r^{\mathbf{A}}(x, y, u) = v'$ and we need to prove that $v' = v$. The algebra $\mathbf{A}$ is Abelian (either it is a singleton, or $k^3 \geq 2$, hence $\mathbf{A}^2$ is Hamiltonian) and so we have that the equality

$$f^{\mathbf{A}}(x, z_1, \ldots, z_n) = u = f^{\mathbf{A}}(x, z_1^{x,u}, \ldots, z_n^{x,u})$$

implies, by the term condition, that

$$v = f^{\mathbf{A}}(y, z_1, \ldots, z_n) = f^{\mathbf{A}}(y, z_1^{x,u}, \ldots, z_n^{x,u}) = v'.$$

Thus $\mathsf{V}(\mathbf{A})$ is indeed Hamiltonian.

To construct the promised examples, let n be an integer greater than or equal to 4, let

$$A = \{0, d, d_1, \ldots, d_n, \hat{d}_1, \ldots, \hat{d}_n\},$$

and let D and $\hat{D}$ be the subsets $\{d_1, \ldots, d_n\}$ and $\{\hat{d}_1, \ldots, \hat{d}_n\}$ of A, respectively. Denote by λ the unary function on A that maps $\{d\} \cup \hat{D}$ to $\{d\}$ and everything else to 0. Define the unary function μ by

$$\mu(d) = \mu(0) = \mu(d_n) = \mu(\hat{d}_n) = 0, \quad \mu(d_i) = \mu(\hat{d}_i) = d_{i+1} \qquad (1 \leq i < n).$$

Let R be the set of all unary functions ρ on A such that

a) $\rho(D) \subset D \cup \{0\}$,

b) $\rho(d) = \rho(0) = 0$,

c) $\rho(d_i) = \rho(\hat{d}_i)$ for all $i \leq n$,

d) the size of $\rho(D) \cap D$ is at most $n - 2$.

Define the binary operation $x + y$ on A by:

$$x + y = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ \hat{d}_i & \text{if } \{x, y\} = \{d, d_i\} \\ d_i & \text{if } \{x, y\} = \{d, \hat{d}_i\} \\ 0 & \text{otherwise} \end{cases} .$$

Finally, let $\mathbf{A}$ be the algebra with universe A and set of basic operations

$$\{\underline{0}, \lambda(x) + \mu(y)\} \cup \{\lambda(x) + \rho(y) \mid \rho \in R\},$$

where $\underline{0}$ stands for the constant zero map. We will show that $\mathbf{A}$ satisfies the desired conditions.

First observe that for $\rho \in R$ we have $\rho\mu$, $\mu\rho \in R$, $\lambda\rho = \rho\lambda = \underline{0}$ and $\mu\mu \in R$. Using the definitions of R, λ, μ and $+$ it can easily be verified that all term operations of $\mathbf{A}$ (except for the projections) have the form

$$\alpha(s) + \beta(t), \qquad \alpha \in \{\underline{0}, \lambda\}, \quad \beta \in \{\underline{0}, \mu\} \cup R,$$

where s and t are not necessarily different variables. Thus every essentially binary term operation of $\mathbf{A}$ is a basic operation, and there are no term operations of higher essential arity.

To finish the proof of the Theorem it is sufficient to show that $\mathbf{A}^{n-2}$ is strongly Hamiltonian, but $\mathsf{V}(\mathbf{A})$ is not Hamiltonian (see the remarks after Definition 1.8).

The second statement follows immediately from the description of the term operations of $\mathbf{A}$ stated above. Indeed, suppose that $r(x, y, z)$ is a ternary Klukovits term operation for the basic operation $f(x, z) = \lambda(x) + \mu(z)$. Since f depends on both of its variables, r must depend on y and z. Hence r is not a projection, and so $r(x, y, z) = \alpha(y) + \beta(z)$ for suitable coefficients α and β. By substituting 0 for y and z, respectively, we obtain that $\alpha = \lambda$ and hence $\beta\mu = \mu$. None of the elements $\beta \in \{\underline{0}, \mu\} \cup R$ satisfy this latter condition. Hence $\mathsf{V}(\mathbf{A})$ is not Hamiltonian.

To show that $\mathbf{A}^{n-2}$ is strongly Hamiltonian, let $f(x, z)$ be a term of $\mathbf{A}$ and $\bar{a}, \bar{b}, \bar{c} \in A^{n-2}$. We have to find a binary term r of $\mathbf{A}$ satisfying

$$r^{\mathbf{A}^{n-2}}(\bar{b}, f^{\mathbf{A}^{n-2}}(\bar{a}, \bar{c})) = f^{\mathbf{A}^{n-2}}(\bar{b}, \bar{c}).$$

If $f^{\mathbf{A}}$ is essentially unary, or constant, then r can be chosen to be a projection. Thus we are left to consider the case where $f(x, z) = \alpha(x) + \beta(z)$, where α and β are not $\underline{0}$. If $\beta = \lambda$, then it can be checked that the identity

$$f(y, f(x, z)) \approx f(y, z)$$

holds in $\mathbf{A}$, so we can take $r = f$. Otherwise we have $\alpha = \lambda$ and $\beta \in \{\mu\} \cup R$. For the components $c_1, \ldots, c_{n-2}$ of $\bar{c}$, let U be the subset $\{\beta(c_1), \ldots, \beta(c_{n-2})\}$ of $D \cup \{0\}$. Pick an element $\rho \in R$ satisfying $\rho(u) = u$ for all $u \in U$. This can be done by the definition of R, since $U \cap D$ has at most $n - 2$ elements. Let $r(x, y) = \lambda(x) + \rho(y)$. Then the desired equality above is clearly satisfied. ∎

Remark 3.2 If $\mathbf{A}$ is a finite algebra of size k, then $\mathsf{V}(\mathbf{A})$ is strongly Abelian if and only if $\mathbf{A}^{k^2}$ is strongly Hamiltonian. The proof is similar to the one just given.

4 Conclusion

There are numerous consequences of the Hamiltonian property that are very important in universal algebra. It is conjectured that for every finitely generated variety, either there is a finite upper bound to the size of all subdirectly irreducible members of the variety (i.e., the variety is residually $< n$ for some natural number n), or there are arbitrarily large infinite subdirectly irreducible algebras in it (i.e., the variety is residually large). This statement is considered to be one of the most important unsolved problems in universal algebra, and is known as the RS-conjecture. This conjecture is due to McKenzie and arose from the following question raised by Quackenbush in [19]: Does a finitely generated variety which contains arbitrarily large finite subdirectly irreducibles also contain an infinite one? An excellent survey of the current status of the RS-conjecture can be found in [16].

By a result of Baldwin and Berman [1], the RS-conjecture holds for varieties having definable principal congruences (DPC). A variety is defined to have DPC if there exists a four variable first order formula in the language of the variety expressing that the first two variables are congruent modulo the principal congruence generated by the second two variables. In such varieties, the class of subdirectly irreducible algebras is elementary.

In the same paper it is shown that every locally finite variety with the congruence extension property (CEP) has DPC. An algebra $\mathbf{A}$ has the CEP if given any subalgebra $\mathbf{B}$ and congruence θ on $\mathbf{B}$, there is a congruence γ on $\mathbf{A}$ such that $\gamma|_B = \theta$. Finally, every Hamiltonian variety has CEP by Kiss [10].

COROLLARY 4.1 *Let $\mathcal{V}$ be a locally finite Abelian variety. Then $\mathcal{V}$ has CEP, DPC, and either $\mathcal{V}$ is residually $< n$ for some natural number n, or $\mathcal{V}$ is residually large.*

We now present the example promised at the end of the Introduction.

Example 4.2 There exists a five element algebra $\mathbf{A}$ such that $\mathsf{HS}(\mathbf{A})$ is Abelian, but $\mathbf{A}$ is not Hamiltonian.

PROOF. Let $\{0, a, b, c\}$ be the Klein–group written additively and u an additional element. Define unary functions on the set $A = \{0, a, b, c, u\}$ by the following table.

	f	g'	g	h
0	0	0	0	0
a	0	a	b	0
b	b	a	b	0
c	b	0	0	0
u	0	0	0	c

Let the algebra $\mathbf{A}$ have underlying set A, and basic operations constant 0, f, g, and t, where t is defined by

$$t(x, y, z) = g'(x) + g'(y) + h(z) .$$

Note that the range of all these unary functions is contained in the Klein–group, and therefore t is well-defined. The only nontrivial congruence of $\mathbf{A}$ has two blocks: singleton $\{u\}$ and all the other elements. Thus $\mathbf{A}$ is not Hamiltonian, because $\{0, a, b\}$ is a subuniverse, but not a congruence block. The verification of the fact that $\mathsf{HS}(\mathbf{A})$ is Abelian is left to the reader (it is of great help that all the operations are linear). ∎

We would like to thank Ralph McKenzie and Keith Kearnes for their helpful remarks while this work was in progress.

References

[1] J. Baldwin and J. Berman. The number of subdirectly irreducible algebras in a variety. *Algebra Universalis*, 5:379–389, 1975.

[2] J. Baldwin and R. McKenzie. Counting models in universal Horn classes. *Algebra Universalis*, 15:359–384, 1982.

[3] J. Berman and R. McKenzie. Clones satisfying the term condition. *Discrete Mathematics*, 52:7–29, 1984.

[4] S. Burris and R. McKenzie. *Decidability and Boolean Representations*, volume 246 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1981.

[5] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra.* Springer-Verlag, 1981.

[6] B. Csákány. Abelian properties of primitive classes of universal algebras. *Acta. Sci. Math. Szeged*, 25:202–208, 1964.

[7] R. Freese and R. McKenzie. *Commutator Theory for Congruence Modular Varieties*, volume 125 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 1987.

[8] C. Herrmann. Affine algebras in congruence modular varieties. *Acta Sci. Math. (Szeged)*, 41:119–125, 1979.

[9] D. Hobby and R. McKenzie. *The Structure of Finite Algebras*, volume 76 of *Contemporary Mathematics.* American Mathematical Society, 1988.

[10] E. Kiss. Each Hamiltonian variety has the congruence extension property. *Algebra Universalis*, 12:395–398, 1981.

[11] E. Kiss and P. Pröhle. Problems and results in tame congruence theory. *Algebra Universalis*, 29:151–171, 1992.

[12] E. Kiss and M. Valeriote. Strongly Abelian varieties and the Hamiltonian property. *The Canadian Journal of Mathematics*, 43:331–346, 1991.

[13] L. Klukovits. Hamiltonian varieties of universal algebras. *Acta. Sci. Math.*, 37:11–15, 1975.

[14] R. McKenzie. On minimal, locally finite varieties with permuting congruence relations. preprint, never published, 1978.

[15] R. McKenzie. Finite forbidden lattices. In *Universal Algebra and Lattice Theory*, volume 1004 of *Springer Lecture Notes.* Springer-Verlag, 1983.

[16] R. McKenzie. Residually small varieties revisited. In *Lattices, Semigroups and Universal Algebra (Proceedings of the Lisbon 1988 Conference)*, pages 185–190. Plenum Press, 1990.

[17] R. McKenzie. Congruence extension, Hamiltonian and Abelian properties in locally finite varieties. *Algebra Universalis*, 28:589–603, 1991.

[18] R. McKenzie, G. McNulty, and W. Taylor. *Algebras, Lattices, Varieties Volume 1.* Wadsworth and Brooks/Cole, Monterey, California, 1987.

[19] R. Quackenbush. Equational classes generated by finite algebras. *Algebra Universalis*, 1:265–266, 1971.

[20] W. Lampe R. Freese and W. Taylor. Congruence lattices of algebras of fixed similarity type, I. *Pacific Journal of Math.*, 82:59–68, 1979.

[21] K. Shoda. Zur theorie der algebraischen erweiterungen. *Osaka Math. Journal*, 4:133–143, 1952.

[22] M. Valeriote. Finite simple Abelian algebras are strictly simple. *Proc. of the Amer. Math. Soc.*, 108:49–57, 1990.

[23] H. Werner. Congruences on products of algebras and functionally complete algebras. *Algebra Universalis*, 4:99–105, 1974.

Eötvös Lóránd University, Department of Algebra and Number Theory, 1088 Budapest, Múzeum krt. 6–8, Hungary.

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1.