Tolerances as congruence images

Conference on Universal Algebra and Lattice Theory

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Tolerance images

Tolerance: compatible, reflexive, symmetric relation. If $\varphi : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism, and T is a tolerance of $\mathbf{A}$, then the image of T under φ, $\underline{\varphi(T)} = \{(\varphi(a), \varphi(b)) \, : \, (a, b) \in \mathbf{A}\}$ is a tolerance of $\mathbf{B}$.

In particular, the image of every congruence is a tolerance.

Problem
Characterize all varieties in which *every tolerance is a homomorphic image of a congruence.* Name: TImC.

Motivating example
The variety of *all lattices* has TImC (Czédli, Grätzer).

Congruence permutable varieties: every tolerance is a congruence.

Linear identities

Linear identity: every variable occurs at most once on each side.

Theorem (Chajda, Czédli, Halaš, Lipparini)
Every variety defined by linear identities has TImC.

Examples

- All algebras of a given similarity type.
- All semigroups.
- All commutative semigroups.

Corollary
Every tolerance is a homomorphic image of a congruence (of an algebra not necessarily in the same variety).

A Mal'tsev-like condition

Condition $M(n)$
For any pair (f,g) of $2n$-ary terms such that the identity

$$f(x_0,x_0,\ldots,x_{n-1},x_{n-1}) \approx g(x_0,x_0,\ldots,x_{n-1},x_{n-1})$$

holds in $\mathcal{V}$, there exists a $4n$-ary term h such that the identities

$$f(x_0,y_0,\ldots,x_{n-1},y_{n-1}) \approx \approx h(x_0,y_0,x_0,y_0,\ldots,x_{n-1},y_{n-1},x_{n-1},y_{n-1})$$

$$g(x_0,y_0,\ldots,x_{n-1},y_{n-1}) \approx \approx h(y_0,x_0,x_0,y_0,\ldots,y_{n-1},x_{n-1},x_{n-1},y_{n-1})$$

also hold in $\mathcal{V}$.
<u>Pattern:</u> $f(xx)=g(xx)$ implies $h(xyxy)=f(xy)$ and $h(yxxy)=g(xy)$.

Theorem (Czédli, Kiss)
A variety satisfies TImC iff it satisfies $M(n)$ for *every* $n \geq 1$.

<u>Remark:</u> No finite set of conditions $M(n)$ suffices.

Lattice varieties

Corollary
Every variety of lattices has TImC.

Proof
If $f(\ldots,x,x,\ldots) \approx g(\ldots,x,x,\ldots)$ is a lattice identity, then let

$$h(\ldots,x,y,u,v,\ldots) = = f(\ldots,x\wedge u,y\wedge v,\ldots)\vee g(\ldots,y\wedge u,x\wedge v,\ldots)\,.$$

Then we have

$$h(\ldots,x,y,x,y,\ldots) = = f(\ldots,x\wedge x,y\wedge y,\ldots)\vee g(\ldots,y\wedge x,x\wedge y,\ldots)\,,$$

which is $f(\ldots,x,y,\ldots)$, since f is monotone, and similarly,

$$h(\ldots,y,x,x,y,\ldots) = = f(\ldots,y\wedge x,x\wedge y,\ldots)\vee g(\ldots,x\wedge x,y\wedge y,\ldots)\,,$$

which is $g(\ldots,x,y,\ldots)$. Thus $M(n)$ holds. □

Further examples

Positive results
The following varieties satisfy $M(n)$ for all n (so have TImC).

- The variety of *semilattices.*
- All algebras of a given similarity type (new proof).
- All varieties of *unary* algebras.

Negative results: Lattices are *idempotent* algebras: $t(x, x, \ldots, x) = x$ for every term; have a *majority* term: $m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x$.

Example
There exists an idempotent variety with a majority term (generated by a 3-element algebra), which fails TImC.

Rules out possible generalizations.

Other varieties without TImC

Theorem
If a congruence n-permutable variety has TImC, then it is congruence permutable.

Proof: by applying $M(n)$ to the Mal'tsev condition discovered by Hagemann and Mitschke.

Problems: discover positive and negative examples

- Which important semigroup varieties have TImC?
- Apply $M(n)$ to other famous Mal'tsev conditions, as above.

Preprint
Czédli, Kiss: *Varieties whose tolerances are homomorphic images of their congruences,* http://arxiv.org/pdf/1204.2228.pdf.