Growth rates of solvable algebras

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Growth rate

The growth rate of a finite algebra $\mathbf{A}$ is the function $d_{\mathbf{A}}(n) =$ *the least size of a generating set* for $\mathbf{A}^n$.

Examples
$\mathbf{A}$ is a module over a ring. Then $d_{\mathbf{A}}(n) = \Theta(n)$ (linear).
Reason: The size of a basis in a vector space $\mathbf{F}^n$ is n.

$\mathbf{A}$ is a Boolean algebra. Then $d_{\mathbf{A}}(n) = \Theta\big(\log(n)\big)$ (logarithmic). The same holds if $\mathbf{A}$ is a simple nonabelian group.
Reason: all finitary functions on $\mathbf{A}$ are polynomials.

$\mathbf{A}$ is a unary algebra. Then $d_{\mathbf{A}}(n) = 2^{\Theta(n)}$ (exponential).
Reason: The free algebras over $\mathbf{A}$ have polynomially bounded size.

Wiegold dichotomy

Theorem (J. Wiegold, 1974)
$\mathbf{G}$ is a finite group. If $\mathbf{G}$ has a nontrivial abelian factor group, then $d_{\mathbf{G}}$ is linear. Otherwise (that is, if $\mathbf{G}$ is perfect) $d_{\mathbf{G}}$ is logarithmic.

Remarks

- If $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$, then $d_{\mathbf{B}}(n) \leq d_{\mathbf{A}}(n)$. So if $\mathbf{G}$ has an abelian factor, then $d_{\mathbf{G}}$ is at least linear.
- If $\mathbf{B}$ is an expansion of $\mathbf{A}$, then $d_{\mathbf{B}}(n) \leq d_{\mathbf{A}}(n)$. The richer the structure, the smaller the growth rate.

Wiegold-dichotomy holds for *Maltsev algebras* (see later).

Motivating problem
What are the possible growth rates of finite algebras?

Pointed cube terms

Example: Maltsev-term ($x_1 x_2^{-1} x_3$ in groups).

$$m\begin{pmatrix} x & y & y \\ y & y & x \end{pmatrix} \approx \begin{pmatrix} x \\ x \end{pmatrix},$$

witnesses that $m(x_1, x_2, x_3)$ is a 3-ary, 0-pointed, 2-cube term.

If Σ is a set of identities in a language $\mathcal{L}$, then an $\mathcal{L}$-term $F(x_1, \ldots, x_m)$ is a *p-pointed, k-cube term* for the variety axiomatized by Σ if there is a $k \times m$ matrix M consisting of variables and p distinct constant symbols, with every column of M containing a symbol different from x, such that

$$\Sigma \models F(M) \approx \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix}.$$

Growth restrictions imposed by identities

Theorem (KKSz)
Let $\mathbf{A}$ be an algebra with an m-ary, $p \geq 1$-pointed, k-cube term, with at least one constant symbol appearing in the cube identities. If $\mathbf{A}^{p+k-1}$ is finitely generated, then all finite powers of $\mathbf{A}$ are finitely generated and $d_{\mathbf{A}}(n)$ is bounded above by a polynomial of degree at most $\log_w(m)$, where $w = 2k/(2k-1)$.

- There exist finite algebras with pointed cube terms whose growth rate is $\sim$ to a polynomial of any prescribed degree.
- The growth rate of any algebra with a pointed cube term arises as the growth rate of an algebra without a pointed cube term.
- If a *basic* Σ does not entail the existence of a pointed cube term, then Σ imposes no restriction on growth rates.
  "Basic" identity: at most one operation symbol on both sides.

General Wiegold dichotomy

If $\mathbf{A}$ has a 0-pointed cube term, then it generates a congruence modular variety. We say that $\mathbf{A}$ is *perfect*, if $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 1_{\mathbf{A}}$ (in the sense of the modular commutator). That is, $\mathbf{A}$ is perfect iff it has no nontrivial abelian factor algebras.

Theorem (KKSz)
Suppose that an algebra $\mathbf{A}$ has a 0-pointed, k-cube term and $\mathbf{A}^k$ is finitely generated.

- $\mathbf{A}$ perfect $\implies d_{\mathbf{A}}(n) = O\big(\log(n)\big)$.
  $\mathbf{A}$ imperfect $\implies d_{\mathbf{A}}(n) = O(n)$.
- Suppose that $\mathbf{A}$ is finite. $\mathbf{A}$ perfect $\implies d_{\mathbf{A}}(n) = \Theta\big(\log(n)\big)$.
  $\mathbf{A}$ imperfect $\implies d_{\mathbf{A}}(n) = \Theta(n)$.

The proof uses a probabilistic argument of independent interest.

Abelianness properties

By R. McKenzie and D. Hobby in tame congruence theory:
If $\alpha, \beta, \delta \in \mathrm{Con}(\mathbf{A})$, then α centralizes β modulo δ, that is, $C(\alpha, \beta; \delta)$ holds iff for all polynomials t of $\mathbf{A}$ we have

$$(\forall \mathbf{a} \equiv_\alpha \mathbf{b})(\forall \mathbf{c} \equiv_\beta \mathbf{d}) \quad t(\mathbf{a}, \mathbf{c}) \equiv_\delta t(\mathbf{a}, \mathbf{d}) \implies t(\mathbf{b}, \mathbf{c}) \equiv_\delta t(\mathbf{b}, \mathbf{d}).$$

The *commutator:* $[\alpha, \beta] = \bigwedge\{\delta \in \mathrm{Con}(\mathbf{A}) : C(\alpha, \beta; \delta) \text{ holds}\}$.

$\mathbf{A}$ is *abelian* if $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$ (that is, $C(1_{\mathbf{A}}, 1_{\mathbf{A}}; 0_{\mathbf{A}})$ holds).
Homomorphic images of abelian algebras are not always abelian.

$\mathbf{A}$ is *solvable,* if there is a chain of congruences $0_{\mathbf{A}} = \theta_0 < \theta_1 < \ldots < \theta_n = 1_{\mathbf{A}}$ such that each θ_{i+1}/θ_i is abelian. Can be expressed with the commutator the same way as for groups.
Homomorphic images, direct products and subalgebras of finite solvable algebras are solvable. A finite algebra is solvable iff only the types **1** and **2** of tame congruence theory occur in it.

Nilpotence

Developed by K. Kearnes:
When $\alpha \in \mathrm{Con}(\mathbf{A})$ define $(\alpha]^1 = [\alpha)^1 = \alpha$ and

$$(\alpha]^{k+1} = [\alpha, (\alpha]^k], \quad [\alpha)^{k+1} = [[\alpha)^k, \alpha].$$

If $(\alpha]^{n+1} = 0$, then α is *n-step left nilpotent*, if $[\alpha)^{n+1} = 0$, then α is *n-step right nilpotent*.
Right nilpotent congruences are left nilpotent in finite algebras.
Left nilpotence implies the following condition:

$$C(1_{\mathbf{A}}, N^2; \delta) \text{ holds whenever } \delta \prec \theta \text{ and } N \text{ is a } \langle \delta, \theta \rangle\text{-trace.} \qquad (\dagger)$$

(Here N^2 is considered as a binary relation, and centrality is defined naturally).
This condition is still stronger than solvability.

Theorem (K. Kearnes)
Homomorphic images of finite abelian algebras are right nilpotent.

The Hamiltonian property

$\mathbf{A}$ is *Hamiltonian:* every subalgebra is a congruence block. *quasi-Hamiltonian:* every maximal subalgebra is a congruence-block.

Theorem (E. W. Kiss, M. Valeriote)
A locally finite variety is abelian iff it is Hamiltonian.

Wielandt: A finite group is quasi-Hamiltonian (that is, every maximal subgroup is normal) iff it is nilpotent.

Theorem (K. Kearnes)
If a finite algebra $\mathbf{A}$ satisfies (†), then it is quasi-Hamiltonian. A variety generated by a finite left nilpotent algebra is quasi-Hamiltonian. Conversely, if $\mathbf{A}^2$ is quasi-Hamiltonian, then $\mathsf{V}(\mathbf{A})$ is quasi-Hamiltonian, and its finite members satisfy (†).

Strongly abelian algebras

An algebra $\mathbf{A}$ is strongly abelian, if for all polynomials t we have

$$(\forall \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) \quad t(\mathbf{a}, \mathbf{c}) = t(\mathbf{b}, \mathbf{d}) \implies t(\mathbf{e}, \mathbf{c}) = t(\mathbf{e}, \mathbf{d}).$$

Let $\mathbf{A}$ be a nontrivial finite algebra and let $\mathbf{B}$ be a nontrivial homomorphic image of $\mathbf{A}^k$ for some k.

- If $\mathbf{B}$ is strongly abelian, then $d_{\mathbf{A}}(n) = 2^{\Theta(n)}$ (exponential).
- If $\mathbf{B}$ is abelian, then $d_{\mathbf{A}}(n) = \Omega(n)$ (at least linear).

This holds, because the free algebras in the first case have polynomially bounded size, and in the second case their size is in $2^{O(n)}$ by a result of J. Berman and R. McKenzie.

Each simple factoralgebra of a finite solvable algebra $\mathbf{A}$ is either abelian or strongly abelian, so $d_A(n)$ is at least linear.

The hierarchy of abelianness properties

(1) **A** is solvable.

(2) **A** is (left) nilpotent.

(3) **A** is abelian.

(4) **A** is a subdirect product of simple abelian algebras.

(5) **A** generates an abelian variety.

We have (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (3). No other implications hold (except the formal consequences).

We prove that *stronger abelianness properties* yield a *closer relationship between various growth-restricting conditions.*
Example: Both (5) and (4) imply that the growth rate is non-exponential iff **A** has a Maltsev term (in which case the growth rate is linear), but (2) does not.

The hierarchy of growth-restricting conditions

(i) **A** has a Maltsev polynomial.

(ii) **A** has a pointed cube polynomial.

(iii) **A** is a spread of its type **2** minimal sets (see later).

(iv) $d_{\mathbf{A}}(n) \in O(n)$.

(v) $d_{\mathbf{A}}(n) \notin 2^{\Omega(n)}$.

(vi) $\mathbf{A}^n$ has no nontrivial strongly abelian factor (for all n).

We have (i)$\Rightarrow$(iv), (i)$\Rightarrow$(ii)$\Rightarrow$(v), and (iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi). No other implications hold for general finite algebras.

Open problem
Is the growth rate of each finite solvable **A** linear or exponential?

True if **A** is nilpotent; would follow from (vi)$\Rightarrow$(iii) for solvable **A**.

Spreads

Definition
Let $\mathbf{A}$ be an algebra and $\mathcal{U}$ a collection of subsets of A. A subset $S \subseteq A$ is a *spread* with respect to $\mathcal{U}$ if there exists a polynomial p of $\mathbf{A}$ and (not necessarily distinct) elements $U_1, \ldots, U_k \in \mathcal{U}$ such that $p(U_1, \ldots, U_k) = S$.

Claim
If a finite algebra $\mathbf{A}$ is a spread of a family of subsets on which the induced algebras have Maltsev polynomials (like type **2** minimal sets), then the growth rate of $\mathbf{A}$ is at most linear.

Theorem (KKSz)
If $\mathbf{A}$ is a finite solvable algebra with a Maltsev polynomial, then $\mathbf{A}$ is a spread of its type **2** minimal sets.

Solvable algebras

Theorem (KKSz)
Let $\mathbf{A}$ be a finite solvable algebra that has a pointed cube term.
Then $d_{\mathbf{A}}(n) = \Theta(n)$.

Tool used: a new characterization of solvability.

Let $\mathbf{A}$ be an algebra and p an idempotent polynomial of $\mathbf{A}$. The *translation-digraph* $\mathbf{Tr}(p)$ on A has directed edges
$(c, c') = \big(p(c, c, \ldots, c), p(c, \ldots, c, d, c, \ldots, c)\big)$, where $c, d \in A$.

Theorem (KKSz)
A finite algebra $\mathbf{A}$ is solvable if and only if for every neighborhood U of $\mathbf{A}$, and every idempotent polynomial p of the induced algebra $\mathbf{A}|_U$, the directed graph $\mathbf{Tr}(p)$ is strongly connected.

Nilpotent algebras

Theorem (KKSz)
A finite left nilpotent algebra has a Maltsev polynomial iff it has a pointed cube polynomial. Hence a finite abelian algebra has a pointed cube polynomial iff it is affine (so has a Maltsev-term).

Theorem (KKSz)
If $\mathbf{A}$ is a finite, left nilpotent algebra, and $\mathbf{A}^{|A|}$ does not have a nontrivial strongly abelian quotient algebra, then $\mathbf{A}$ is a spread of its type **2** minimal sets (hence linear).

The proof uses the quasi-Hamiltonian property for the subalgebras of $\mathbf{A}^{|A|}$.

Abelian varieties

Theorem (KKSz)
Let $\mathbf{A}$ be an algebra, which is a spread of subsets whose induced algebras are affine. Then the following hold.

- If $\mathsf{H}(\mathbf{A}^2)$ is abelian, then there is an abelian group operation on A that is compatible with all operations of $\mathbf{A}$, and preserves all congruences of $\mathbf{A}$.
- If the variety $\mathsf{V}(\mathbf{A})$ generated by $\mathbf{A}$ is abelian, then $\mathbf{A}$ is affine.

Examples
An 8-element quasi-affine algebra shows that in the second statement the assumption that $\mathsf{V}(\mathbf{A})$ is abelian cannot be dropped.

Another 8-element abelian algebra shows that in the first statement it is not sufficient to assume only that $\mathsf{H}(\mathbf{A})$ is abelian.

Semisimple algebras

Theorem (KKSz)
Let $\mathbf{A}$ be a finite solvable algebra and β the intersection of all maximal congruences of $\mathbf{A}$. If the growth rate of $\mathbf{A}/\beta$ is linear, then $\mathbf{A}/\beta$ has a Maltsev polynomial. In particular, if $\mathbf{A}$ is (linear, and) a direct product of simple abelian algebras, then $\mathbf{A}$ is Maltsev.

The proof shows that $\mathbf{A}/\beta$ is a direct product, and not just a subdirect product of simple abelian algebras.

Example
There exist a 16-element algebra that is a direct product of two, 4-element affine (hence abelian, Maltsev) algebras, has a linear growth rate, but does not have a Maltsev polynomial.

Summary: arbitrary

(i) $\mathbf{A}$ has a Maltsev polynomial.

(ii) $\mathbf{A}$ has a pointed cube polynomial.

(iii) $\mathbf{A}$ is a spread of its type $\mathbf{2}$ minimal sets.

(iv) $d_{\mathbf{A}}(n) \in O(n)$.

(v) $d_{\mathbf{A}}(n) \notin 2^{\Omega(n)}$.

(vi) $\mathbf{A}^n$ has no nontrivial strongly abelian factor (for all n).

All are equivalent if $\mathbf{A}$ is semisimple or if $\mathsf{V}(A)$ is abelian.

$$\begin{array}{ccccccc} & & (i) & \Longrightarrow & (ii) & & \\ & & \Downarrow & & \Downarrow & & \\ (iii) & \Longrightarrow & (iv) & \Longrightarrow & (v) & \Longrightarrow & (vi). \end{array}$$

For arbitrary finite algebras

Summary: solvable

(i) $\mathbf{A}$ has a Maltsev polynomial.

(ii) $\mathbf{A}$ has a pointed cube polynomial.

(iii) $\mathbf{A}$ is a spread of its type $\mathbf{2}$ minimal sets.

(iv) $d_{\mathbf{A}}(n) \in O(n)$.

(v) $d_{\mathbf{A}}(n) \notin 2^{\Omega(n)}$.

(vi) $\mathbf{A}^n$ has no nontrivial strongly abelian factor (for all n).

All are equivalent if $\mathbf{A}$ is semisimple or if $\mathsf{V}(A)$ is abelian.

$$\begin{array}{ccccccc} & & (i) & \Longrightarrow & (ii) & & \\ & \swarrow\!\!\swarrow & & \swarrow\!\!\swarrow & & & \\ (iii) & \Longrightarrow & (iv) & \Longrightarrow & (v) & \Longrightarrow & (vi). \end{array}$$

For finite, solvable algebras

Summary: nilpotent

(i) $\mathbf{A}$ has a Maltsev polynomial.

(ii) $\mathbf{A}$ has a pointed cube polynomial.

(iii) $\mathbf{A}$ is a spread of its type $\mathbf{2}$ minimal sets.

(iv) $d_{\mathbf{A}}(n) \in O(n)$.

(v) $d_{\mathbf{A}}(n) \notin 2^{\Omega(n)}$.

(vi) $\mathbf{A}^n$ has no nontrivial strongly abelian factor (for all n).

All are equivalent if $\mathbf{A}$ is semisimple or if $\mathsf{V}(A)$ is abelian.

$$
\begin{array}{ccccccc}
 & & (i) & \Longleftrightarrow & (ii) & & \\
 & \swarrow & & \swarrow & & & \\
(iii) & \Longleftrightarrow & (iv) & \Longleftrightarrow & (v) & \Longleftrightarrow & (vi).
\end{array}
$$

For finite, left nilpotent algebras

Open problems

Is there a finite algebra $\mathbf{A}$ such that $d_{\mathbf{A}}(n) \notin \Omega(n)$ and $d_{\mathbf{A}}(n) \notin O(\log(n))$? That is, whose growth rate is between logarithmic and linear? Open for 2-element partial algebras, too.

Is it true that a finite algebra with a 2-sided unit for some binary term has logarithmic or linear growth? (Note that the identities $x * 1 = 1 * x = x$ show that $*$ is a 1-pointed 2-cube term.)

Does (ii)⇒(iii) hold for finite solvable algebras?

Which of the true implications (iii)⇒(iv)⇒(v)⇒(vi) can be reversed for finite solvable algebras? In particular, is the growth rate of a finite solvable algebra always linear or exponential?

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