The following is a proof of the Waring's formula using formal power series. We will work with formal power series in indeterminate z with coefficients in the ring $\mathbb{Q}[x_1, \ldots, x_n]$. We also need the following equality

$$-\log(1-z) = \sum_{j=1}^{\infty} \frac{z^j}{j}.$$

Taking log on both sides of

$$1 - \sigma_1 z + \ldots + (-1)^n \sigma_n z^n = \prod_{m=1}^{n} (1 - x_m z),$$

we get

$$\log(1 - \sigma_1 z + \ldots + (-1)^n \sigma_n z^n) = \sum_{m=1}^{n} \log(1 - x_m z), \tag{1}$$

Waring's formula will follow by comparing the coefficients on both sides.

The right hand side of the above equation equals

$$\sum_{m=1}^{n} \sum_{j=1}^{\infty} \frac{x_m^j}{j} z^j$$

or

$$\sum_{j=1}^{\infty} \left(\sum_{m=1}^{n} x_m^j \right) \frac{z^j}{j}$$

The coefficient of z^k is equal to S_k/k.

On the other hand, the left hand side of (1) can be written as

$$\sum_{j=1}^{\infty} \frac{1}{j} (\sigma_1 z - \sigma_2 z^2 + \ldots + (-1)^{n-1} \sigma_n z^n)^j.$$

For each j, the coefficient of z^k in

$$(\sigma_1 z - \sigma_2 z^2 + \ldots + (-1)^{n-1} \sigma_n z^n)^j$$

is

$$\sum_{i_1, \ldots, i_n} (-1)^{i_2 + i_4 + i_6 + \ldots} \frac{j!}{i_1! \cdots i_n!} \sigma_1^{i_1} \cdots \sigma_n^{i_n},$$

where the summation is extended over all n-tuple $(i_1, \ldots, i_n)$ whose entries are non-negative integers, such that

$$i_1 + i_2 + \ldots + i_n = j$$
$$i_1 + 2i_2 + \ldots + n i_n = k.$$

So the coefficient of z^k in the left hand side of (1) is

$$\sum_{j=1}^{\infty} \sum_{i_1, \ldots, i_n} (-1)^{i_2 + i_4 + i_6 + \ldots} \frac{(j-1)!}{i_1! \cdots i_n!} \sigma_1^{i_1} \cdots \sigma_n^{i_n},$$

or

$$\sum (-1)^{i_2 + i_4 + i_6 + \ldots} \frac{(i_1 + \ldots + i_n - 1)!}{i_1! \cdots i_n!} \sigma_1^{i_1} \cdots \sigma_n^{i_n}.$$

The last summation is over all $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ with non-negative entries such that $i_1 + 2i_2 + \ldots + n i_n = k$.